Let F be a non-archimedean local field, G_F Galois gp

$$1 \longrightarrow I_F \longrightarrow G_F \longrightarrow \hat{\mathbb{Z}} \longrightarrow 1$$

$$1 \longrightarrow I_F \longrightarrow W_F \longrightarrow \mathbb{Z} \longrightarrow 1$$

$$W_F \to \mathbb{R}^\times$$

$$\begin{tikzcd} W_F \arrow[r] \arrow[dr] & F^\times \arrow[r, "\|\cdot\|"] & \mathbb{R}^\times \\ & \mathbb{Z} \arrow[u] & \end{tikzcd} \qquad 1 \mapsto \omega_F$$

Weil-Deligne repsentations

A W-D repr is a triple (ρ, V, n)

(ρ, V) a f.d. repr of $W_F / \mathbb{C}$, smooth, i.e. kernel $\rho \supset$ open subgp of I_F

$n \in \mathrm{End}_{\mathbb{C}}(V)$ nilpotent

s.t. $\rho(x)\, n\, \rho(x)^{-1} = \|x\|\, n$

It is semi-simple if $\rho(\mathrm{Frob}_F)$ is semi-simple

<u>Motivation</u>: Really want something like continuous representation of W_F on $\overline{\mathbb{Q}}_\ell$- v.s.

structure theory of these $\longleftrightarrow$ WD, Fr-semisimple

LLC: $\left\{\begin{array}{l}\text{semi-simple} \\ n\text{-dimensional correspondence}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{fixed smooth repr} \\ \text{of } GL_n(F)\end{array}\right\}$

irr of W_F = irr of G_F up to twist by character

2 ways to build 2 dim'l repr of W_F → Two ways to build repr of WD-gp

Play with n. $p > n$ ⇒ pro p gp acts via characters

Suppose $p > n$ $\boxed{\text{Fact}}$ n-dim'l irrd rep of W_F are induced from 1 dimensional repr. of an index n subgroup.

Index n subgp + 1 dim'l repr = E/F deg n + repr of W_E $\overset{\text{CFT}}{=}$ E/F deg n + $\underbrace{\text{char}}_{\text{of}}$ $E^\times$

$\xrightarrow{\text{LLC}}$ Cuspidal irreducible repr of $GL_n(F)$

For $p \leq n$ there are more repr, e.g. $SL_2(\mathbb{F}_3) \hookrightarrow GL_2(\mathbb{C})$

binary tetrahedral subgroup

is a Galois group of $E/\mathbb{Q}_2$ F quadratic unramified

not an induced representation.

<u>Theorem</u> Let G be a reductive p-adic gp. (V,π) smooth irreducible representation. Then (V,π) is admissible, i.e., for any cpct open subgp $K\subset G$, V^K is finite dimensional.

<u>Ref</u> Renard book : Representations des groupes p-adiques
Ngo's notes : ——— " ———

1. $(V,\pi) \rightsquigarrow$ can embed it in $\mathrm{Ind}_M^G(E,\tau)$, (E,τ) irreducible cuspidal, $P=MN$ parabolic.

2. Any cuspidal irred rep is cpct/center $\overset{H\text{-}C}{\Longrightarrow}$ it is admissible

3. Parabolic induction preserves admissibility.

1. <u>Normalised parabolic induction</u>

$P = MN$ parabolic $\subset G$, $\mathrm{Rep}(M) \xrightarrow{\mathrm{Inf}_M^P} \mathrm{Rep}(P) \xrightarrow{\mathrm{Ind}_P^G} \mathrm{Rep}(G)$

$i_M^G : \mathrm{Rep}(M) \longrightarrow \mathrm{Rep}(G)$ $\quad \sigma \longmapsto \mathrm{Ind}_P^G\left(\mathrm{Inf}_M^P(\sigma)\otimes \Delta_P^{\frac{1}{2}}\right)$

Here $\Delta_P : P \longrightarrow \mathbb{C}^*$ modulus character

$\mu(Sg) = \Delta_P(g) = \mu(S)$ $\forall g\in P$, $S\subset P$ open

Adjoint $\quad \mathrm{Rep}(G) \xrightarrow{\text{res}} \mathrm{Rep}(P) \xrightarrow{J_M^P} \mathrm{Rep}\, M$

(V, π)

$V(N) = \mathrm{Span}\{\pi(n)v - v \mid n \in N\ v \in V\}$

$(V, \pi) \longmapsto (V_N = V/V(N), \pi_N)$ (Left adjoint without normal.)

<u>Define</u> $\quad r_M^G : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(M)$

$\pi \longmapsto J_M^P(\mathrm{Res}_P^G(\pi) \otimes \delta_P^{-\frac{1}{2}})$

<u>Lemma</u> Suppose $P = MN$ is parabolic in G, suppose (E, τ) is admissible smooth rep. of M then $i_M^G(E, \tau)$ is admissible

<u>Proof</u> (E, τ) denotes still the inflated representation

Suppose $K \subset G$ cpct open, let $\Omega = P \backslash G / K$ a finite set

$(W, \rho) := i_M^G(E, \tau)$

$W^K \xrightarrow{l_\Omega} \{ f: \underset{\substack{\uparrow \\ \text{f.d.}}}{\Omega} \to E \mid f(g) \in \underbrace{E^{P \cap gKg^{-1}}}_{\text{f.d}} \ \forall g \in G\}$

$\Rightarrow W^K$ is finite dimensional.

<u>Proposition</u> a) Let (V, π) be smooth irred repn of G there exists a parabolic $P = MN$ s.t. $r_M^G(V, \pi)$ is a cuspidal repn of M

b) Frobenius reciprocity $\Rightarrow$ have the embedding

<u>Proof</u> Choose $P = MN$ a parabolic that is minimal w.r.t $r_M^G(V,\pi) \neq 0$. By def it is cuspidal. $\blacksquare$

Compact modulo center $\Rightarrow$ admissible

$C_c^\infty(G)$ = smooth fcns of compact support

$\mathcal{D}(G)$ = linear maps $C_c^\infty(G) \to \mathbb{C}$ distribution

$\mathcal{D}_c(G) = \{$ continuous linear $C_c^\infty(G) \to \mathbb{C} \}$

compact convergence topology

discrete topology

$\{\varphi_n\} \subset C_c^\infty(G)$ converges to φ iff

$\forall$ compact open subset $K \subset G$ s.t. $\exists N \in \mathbb{N}$

s.t $\varphi_n|K = \varphi|K$ for $n > N$

Any smooth representation gives rise to a $\mathcal{D}_c(G)$-module

$\mathcal{H}(G) = \{ \xi \in \mathcal{D}_c(G)$ smooth w.r.t left translation$\}$

$g \longmapsto \ell_g(\xi) \qquad G \to \mathcal{D}_c(G)$ smooth

~~[illegible]~~ A smooth repr of $G \to \mathcal{H}(G)$-module

A cpct open subgroup K of G defines an element $e_K \in \mathcal{H}(G)$

Fact If (V, π) is a smooth representation of G and $K \subset G$ cpct open subgroup the $v \in V^K$ iff $v = \pi(e_K)v$

$\varphi \in \mathcal{D}_c(G)$, $v \in V$

$$\pi(\varphi)v = \sum_{i=1}^{n} \varphi(1_{K_i}) v_i$$

Let $K \subset G$ cpct open containing the support of φ

$g \longmapsto \pi(g)v$ is locally constant

$\downarrow$ restrict to K

$$\sum 1_{K_i} v_i$$

Prop (E, τ) Compact mod center irred $\Rightarrow$ admissible

Pf Let $K \subset G$ cpct open subgroup. Let $v \in E^K$, $v \neq 0$
Consider $\varphi_{K,v} : G \to E^K$, $g \mapsto \tau(e_K * \delta_g * e_K)v$

Let V = span of image of $\varphi_{K,v} \subset E^K$

1. $V = E^K$, $\mathcal{H}_K = e_K * \mathcal{D}_c(G) * e_K$ acts on V

   E irreducible $\Rightarrow$ E^K irreducible $\mathcal{H}_K$-module
   $\Rightarrow V = E^K$

2. V is finite dimensional. Suppose not, $\exists\, g_1 \cdots$ s.t. $v_i = \varphi_{K,v}(g_i) \in E^K$ are linearly independent

   Let $v^* \in E^*$ s.t. $\langle v^*, v_i \rangle = 1 \quad \forall\, i$

   Set $v' = e_K v^* \in \underset{\substack{\uparrow \\ \text{smooth}}}{E'}$, then $\langle v_i, v' \rangle = 1 \quad \forall i$

   Then the matrix coefficient attached to v, v' does not have compact support.