① Weil gps vs Galois gps

(1.1) class field theory K - local / global field

Defn $C_K = \begin{cases} K^\times & \text{local} \\ \mathbb{A}_K^\times / K^\times & \text{global} \end{cases}$

Thm (Artin reciprocity) There is a homom. of gps called "reciprocity" map

$$\theta_K : C_K \to G_K^{ab} \quad \text{with dense image.}$$

(C_K not profinite, G_K^{ab} profinite)

If L/K finite ext $\theta_{L/K} : C_K / N_{L/K} C_L \xrightarrow{\sim} G(L/K)^{ab}$

$$\alpha \mapsto (\alpha, L/K)$$

If $K \subseteq K'$ separable, the diagram

(functoriality)
$$\begin{array}{ccc} C_K & \longrightarrow & G_K^{ab} \\ \downarrow & & \downarrow \text{transfer} \\ C_{K'} & \longrightarrow & G_{K'}^{ab} \end{array}$$

first gp cohom. (corestriction?)

commutes.

<u>Remarks</u> a) k non-archimedean local,

then $k^\times \cong \mathbb{Z} \times \mathcal{O}_k^\times \cong \underset{\substack{\uparrow \\ \text{roots} \\ \text{of unity}}}{\mathbb{Z} \times \mu(k)} \times \underset{\substack{\uparrow \\ \text{pro-}p\text{-gp}}}{\tilde{P}}$

θ_k is injective, not surjective.

b) If k is a #-field θ_k is surjective but not injective. with kernel C_k° = conn. component.

e.g. $k = \mathbb{Q}$ $C_{\mathbb{Q}}^\circ = \mathbb{R}_{>0}$

c) L/k finite abelian

$$\begin{array}{ccc} \hat{H}^{-2}(G(L/k), \mathbb{Z}) & \xrightarrow[\cup\, u_{L/k}]{\sim} & \hat{H}^0(G_k, C_L) \\ \wr| & & \wr| \\ H_1(G(L/k), \mathbb{Z}) & \xleftarrow[\theta_{L/k}]{\sim} & C_k / N_{L/k} C_L \\ \| & & \\ G(L/k)^{ab} & & \end{array}$$

$\theta_k = \varprojlim_L \theta_{L/k}$

can choose these in a coherent way $\rightarrow$ $u_{L/k} \in H^2(G_k, C_L)$

local: $\underset{\text{Brauer gps.}}{H^2(G_k, L^\times) \subseteq Br(k)} \xrightarrow[\text{inv}]{\sim} \mathbb{Q}/\mathbb{Z}$

(1.2) Weil gps

Defn A Weil gp of K is a triple

$(W_K, \varphi, \{r_L\}_{L/K})$

L/K finite ext.

W_K is a topological gp

$\varphi : W_K \to G_K$ is a cont. hom with dense image

$r_L : C_L \xrightarrow{\sim} W_L^{ab}$ is an iso.

+ axioms. In particular

$$C_L \xrightarrow{\sim} W_L^{ab} \xrightarrow{\varphi} G_L^{ab} \quad \text{is } \theta_L$$

and $W_K \xrightarrow{\sim} \varprojlim W_{L/K}$ $\qquad W_{L/K} = W_K / W_L^c$

not pro-finite. $\qquad$ closure of commutator

Note $W_K / W_L \cong G(L/K)$ $\qquad$ (chosen a separable closure)

<u>Thm</u> (Weil 1951)

(a) A Weil gp exists

(b) Any two are isomorphic. (non-uniquely)

Construction works by constructing the $W_{L/K}$ as extensions $1 \to C_L \to W_{L/K} \to G(L/K) \to 1$.

classified by $u_{L/K}$.

<u>Rmk</u> (a) If k is non-archimedean local,

$$\begin{array}{ccc} W_K & \overset{\varphi}{\hookrightarrow} & G_K \\ \downarrow & & \downarrow \\ \mathbb{Z} & \hookrightarrow & \hat{\mathbb{Z}} \end{array} = \mathrm{Gal}(\bar{k}/k) \quad k\text{-residue field.}$$

Thus $I_K \subseteq W_K$ $\quad W_K/I_K \cong \mathbb{Z}$

and I_K is open in W_K, profinite.

(b) If $K = \mathbb{C}$ then $W_K = \mathbb{C}^\times$ φ trivial

$r_{\mathbb{C}} = \mathrm{id}$.

If $k = \mathbb{R}$ $\quad W_k = \mathbb{C}^\times \cup j\mathbb{C}^\times$

with $\quad j^2 = -1 \quad jcj^{-1} = \bar{c}$

$\varphi(\mathbb{C}^\times) = \{1\}$, $\quad \varphi(j\mathbb{C}^\times) = \{\ \}$ complex conj.

$r_\mathbb{C}$ identity $\qquad r_\mathbb{R} : \mathbb{R}^\times \to W_\mathbb{R}^{ab}$

$-1 \mapsto j W_\mathbb{R}^c$

$0 < x \mapsto \sqrt{x}\, W_\mathbb{R}^c$.

$W_\mathbb{R}^c = \{ \frac{z}{\bar{z}} \mid z \in \mathbb{C}^\times \}$

(c) K # field $\quad$ then φ is surjective

with kernel the connected component.

$\cong \varprojlim_L C_L^0$

<u>Defn</u> Given W_K we define the norm

$\| \cdot \| : W_K \to \mathbb{R}_{>0}$ by

$\|w\| = \| r_K^{-1}(w) \|$

$\quad\quad\quad\quad\quad \underset{C_K}{\cap}$

<u>Rmk</u> (a) K : #-field $\quad \|\cdot\|$ is surjective

$$W_K \cong \mathbb{R} \times W_K'$$

(6) If k is non-achimedean local, then the image of $\|\cdot\|$ is $q^{\mathbb{Z}}$, $q = |\kappa|$

$$W_K = \mathbb{Z} \ltimes W_K'$$

② Representations $\quad M(W_K)$ $\quad$ f.diml, complex cont. repns.

e.g. quasi-charaters $\quad W_K \to \mathbb{C}^\times$

e.g. for $s \in \mathbb{C}$ $\quad \omega_s(w) = \|w\|^s$

$$R(W_K) = M(W_K)^+$$

If k # field, v place, get

$$M(W_K) \xrightarrow{\text{res}} M(W_{K_v})$$

$$R(W_K) \xrightarrow{\text{res}} R(W_{K_v})$$

There is Frobenius reciprocity

$$R(W_K) \rightleftarrows R(W_L) \quad L/K$$

Defn We call a repn ρ of W_K of "Galois type" if it is restricted from G_K $\iff$ $\rho(W_K) \subseteq GL(V)$ is finite.

Lemma If K is non-archimedean local and ρ is irreducible, then $\exists$ σ of Galois type and $s \in \mathbb{C}$ s.t $\rho \cong \sigma \otimes \omega_s$.

Prop $R(W_K)$ is generated as a gp by elts of the form $\mathrm{Ind}_{L/K}(\chi)$ L/K finite, χ a charater

Defn A function λ assigning to L/K $\rho \in M(W_L)$ and element $\lambda(\rho) \in X$ ← abelian gp is called inductive if it is additive and commutes with $\mathrm{Ind}_{L'/L}$.

Ex let k be a #-field

v place, λ inductive over k_v

Define $\lambda_v(\rho) := \prod_{w/v} \lambda(\rho_w)$ is inductive over k.

$\rho \in M(W_L)$

3 L-functions

Let k be a non-archimedean local field

$\pi \in k$ a uniformizer

(3.1) Let $\chi : k^\times \to \mathbb{C}^\times$ be a character

Define $L(\chi) = \begin{cases} (1-\chi(\pi))^{-1} & \text{unramified, i.e. factors through } \mathbb{Z} \\ 1 & \text{else} \end{cases}$

$L(\chi, s) := L(\chi \otimes \omega_s)$ is a meromorphic function without zeros.

Let $I_k \subseteq W_k$ be the inertia subgp choose $\Phi_k \in W_K$ s.t $\|\Phi_k\| = \|\pi_k\|_K$

and similarly for K'/K.

<u>Thm</u> (Artin) The function L assigning to $\rho \in M(W_{K'})$ the meromorphic fn

$L(\rho, s) = L(\rho \otimes \omega_s)$ where $L(\tilde{\rho})$ is defined to be $\det(1 - \Phi_{k'}|_{V^{I_{K'}}})^{-1}$, is

indutive over K, and does the above on charaters.

<u>Remark</u> (a) For a (global) # field K one define L-fns of ρ as product of $L(\rho_v, s)$ v places of K

This will be indutive.

(b) $K = \mathbb{R}, \mathbb{C}$ $\pi^{-s/2}\Gamma(s/2)$, $\pi^{?}\Gamma(?)$

local factors.

ε-factors

Langlands proved existence of a inductive function $\varepsilon(\rho, \cdots)$

Lecture notes
Modular fns in one var.
Deligne

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④ Weil-Deligne gp / repns

Hecke L-fns
Artin L-fns

(4.1) Motives (pure)

cohomology theories, comparison thms

$$\mathrm{Var}/k^{op} \xrightarrow[h(-)]{} \mathcal{M} \xrightarrow{\text{realisation}} \text{(a Tannakian category) over } F \supseteq \mathbb{Q}$$

smooth & proj.

with $\mathbb{Q}$-coefficients

$$h(X) = \bigoplus_{i=0}^{2d} h^i(X)$$

$\mathcal{M}_{hom} \cong \mathrm{Rep}_{\mathbb{Q}}$ (motivic Galois gp)

Morphisms are (basically) correspondences.

Rmk $\mathcal{M}_{hom}$ can be constructed, but it may not be Tannakian.

(Everything works if hom = num) $\Leftrightarrow$ int. with all coho classes

hom equiv to 0

Given a motive, say $h^i(X)$ say over a #-field k, can produce (for v a finite place of k) a continuous G_{k_v}-module $H^i_{et}(X_{\bar{k}_v}, \mathbb{Q}_\ell)$ $v \nmid \ell$

(for each choice of ℓ and each choice $\bar{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$, get an Euler factor $L_v(h^i(X), s)$

This is well-defined under "independent of ℓ" which holds:

- good reduction. (by Deligne's Weil 2, proper & smooth base-change)
- semi-stable reduction (curves, abelian varieties)

Now, given a continuous ℓ-adic repn

$$G_K \to GL(V), \quad V \text{ is an } E\text{-v.s}$$

G_K local

where $E/\mathbb{Q}_\ell$ finite

$(p \neq \ell)$

Let k be local $\quad G_K \xrightarrow{\chi} \mathbb{Z}_\ell$

$k = \mathbb{Q}_p$

Then $\rho = \begin{pmatrix} 1 & \chi \\ & 1 \end{pmatrix}$ is a continuous 2-diml ℓ-adic repn of G_K. not semisimple.

(occurs : elliptic curve semi-stable red, not good red)

<u>Thm</u> (Grothendieck's local monodromy theorem)

k local $\quad p \neq \ell \qquad G_K \supseteq I_K \supseteq P_K$

Let $\rho : G_K \to GL_n(E)$ be a cont. repn. Then there exists an relatively open subgp $U \leq I_K$ and a nilp $N \in M_n(E)$, s.t.

$\forall I \in U \quad \rho(I) = \exp(t_\ell(I) N)$

Here: $\quad t_\ell : \quad I_K \to I_K / P_K \cong \prod_{\ell' \neq p} \mathbb{Z}_{\ell'} \xrightarrow{\text{proj}} \mathbb{Z}_\ell$

$\operatorname{spec} K \subseteq \operatorname{Spec} \mathcal{O}_K \supseteq \operatorname{Spec} \underbrace{k}_{\text{res. field}}$

<u>Pf</u> $\ker(\rho)$ contains an open subgp of $\ker(t_\ell)$

By shrinking U, we can assume that

$\rho|_{U \cap \ker(t_\ell)}$ is trivial.

Now the image of ρ stabilizes a lattice.

so we may assume that $\operatorname{Im} \rho \subseteq GL_n(\mathcal{O}_E)$

Shrinking U, we can arrange

$$\rho(U) \subseteq \{ g \mid \underbrace{g \equiv 1}_{|g-1|<1} \pmod{\ell^2} \} = K_{(2)}$$

Thus $\log \rho(I)$ makes sense for $I \in U$.

$$t_\ell(u) \cong u / u \cap \ker(t_\ell) \xrightarrow{\rho} K_{(2)} \xrightarrow{\log} \ell^2 M_n(\mathcal{O}_E)$$ (14)

$\quad \|$

$\ell^m \mathbb{Z}_\ell$ for some m

Thus, the composite above is of the form $\quad x \mapsto xN \quad$ some $N \in M_n(E)$

<u>Claim</u> $\quad N$ is nilpotent. To see this, let $\Phi \in G_K$ be a lift of the arithmetic Frobenius. Then for $\tau \in I_K$.

$$t_\ell(\Phi \tau \Phi^{-1}) = (\# k) \cdot t_\ell(\tau)$$

$\downarrow$ res. field

$$\Rightarrow \rho(\Phi) N \rho(\Phi)^{-1} = q_K N$$

$\Rightarrow$ all eigenvalues of N are zero.

hence is nilp. (b/c q_K is not a root of unity in E)

Theorem (Deligne)

Let t_L be as above, $W_K \to GL_n(E)$ a ct repn, U, N as in the local monodromy thm. Then the formula

$$\rho_\#(\Phi^m \tau) = \rho(\Phi)^m \rho(\tau) \exp(-t_L(\tau) N)$$

defines a new repn of W_K on E^n, and its isom class does not depend on a choice of Φ.

Rmk 1) $\rho_\# |_{I_K}$ is trivial on U, hence s.s.

2) Therefore, $\rho_\#$ is semi-simple $\Leftrightarrow$ $\rho_\#(\Phi)$ is semi-simple.

Defn Weil-Deligne repn is a pair $(\rho_\#, N)$ where $\rho_\#: W_K \to GL_n(\mathbb{C})$ is continuous and $N \in M_n(\mathbb{C})$ is nilp s.t

$$\rho_\#(\sigma) N \rho_\#(\sigma)^{-1} = \|\sigma\| N \quad \text{for } \sigma \in W_K$$

In particular, Deligne's thm produces a well-defined Weil-Deligne repn out of any continuous ℓ-adic W_K-repn once we choose an isom $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$

<u>Rmk</u> If $\rho_\#(\Phi)$ is s.s, then $\rho_\#(\sigma)$ is s.s $\forall\, \sigma \in W_K$. In this case, we call $(\rho_\#, N)$ "Frob.-s.s"

<u>L-function</u>

$$L((\rho_\#, N), s) := \det\left(1 - q_K^{-s}\, \Phi|_{\ker(N)^{I_K}}\right)^{-1}$$

<u>Ex</u> $\rho = \begin{pmatrix} \chi_{cyc} & t_\ell \\ 0 & 1 \end{pmatrix}$ Tate module of an elliptic curve w/ semistable reduction but not good reduction.

$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$