Oct 22, 2019 Rep theory seminar P. McNamara

§1 Building for p-adic GL_n

F - loc. nonarch. field.

Let V be a finite dim'l v. space $/F$

<u>$GL(V)$ story</u>

A norm on V is a function

$\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ s.t

1) $\|x+y\| \leq \max(\|x\|, \|y\|)$ $\quad x \in V$

$\|\lambda x\| = |\lambda| \, \|x\|$ $\quad \lambda \in F$

$\|x\| = 0 \iff x = 0$

Sometimes you might want to think about

$\alpha = -\log\|\cdot\|$ instead $\alpha : V \to \mathbb{R} \cup \{\infty\}$

<u>Lemma</u> if $\|x\| \neq \|y\|$

then $\|x+y\| = \max(\|x\|, \|y\|)$

<u>Example</u> Let $e_1, \cdots, e_n$ be basis of V

Let $c_1, \cdots, c_n \in \mathbb{R}$

Then $\|\sum \lambda_i e_i\| = \max\left(e^{c_i}|\lambda_i|\right)$ is a norm.

$\alpha(\sum \lambda_i e_i) = \min_i \left(v(\lambda_i) + c_i\right)$ is a norm.

<u>Def</u> The building for $GL(V)$ is

$\mathcal{B} = \{\text{all norms on } V\}$

Each ordered basis of V gives $\mathbb{R}^n \hookrightarrow \mathcal{B}$.

Image is an apartment. (def. of apartment)

<u>Thm</u> Every point lies in an apartment.

(true $\forall$ buildings/gps)

<u>Pf</u> Induct on $\dim V$

Let $W \leq V$ be codim 1 $v \in V \setminus W$

Let $e_1, \cdots, e_{n-1}$ be a basis of W

s.t $\|\sum_{i=1}^{n-1} \lambda_i e_i\| = \max_{i=1}^{n-1} \left(e^{c_i}|\lambda_i|\right)$ by ind.

WLOG. $\{x \in F \mid \|xe_i\| \leq \|v\|\} = \mathcal{O}_F \quad \forall i$

(rescale e_i if necessary)

Let $X = \{v + \sum_{i=1}^{n-1} \zeta_i e_i \mid \zeta_i \in \mathcal{O}_F\}$ is cpt

$\Rightarrow \exists\, e_n \in X$ s.t $\|e_n\| \leq \|x\| \quad \forall x \in X$.

<u>claim</u> $e_1, \cdots, e_n$ is the desired basis of V.

Have to show

$$\|e_n + \sum_{i=1}^{n} \lambda_i e_i\| = \max(\|e_n\|, \|\lambda_i e_i\|)$$

$$= \max(\|e_n\|, \|\sum_{i=1}^{n-1} \lambda_i e_i\|)$$

If we don't have equality, then

$$\|e_n\| = \|\lambda_i e_i\|$$

$$\& \quad \|e_n\| \geq \|\lambda_j e_j\| \quad \forall j \qquad \lambda_j \in \mathcal{O}_F$$

$$e_n + \sum \lambda_i e_i \in V + \sum \mathcal{O}_F e_i = X$$

so $\|e_n + \sum \lambda_i e_i\| \geq \|e_n\|$

Since $\|e_n + \sum \lambda_i e_i\| \le \max(\|e_n\|, \|\sum \lambda_i e_i\|) = \|e_n\|$

have equality $\square$

<u>Q</u> What is stabilizer of a pt ?

(Wrong question for general G, ok for $[G,G]$

Also stabilizer of facet ?)

Let $e_1, \cdots e_n$ be a splitting basis

i.e. $\|\sum \lambda_i e_i\| = \max(|\lambda| \|e_i\|)$

Let $\omega_F \in \mathcal{O}_F$ be uniformizer.

WLOG : $\|e_1\| \ge \|e_2\| \ge \cdots \ge \|e_n\| > \|\omega_F e_1\|$

$g = (g_{ij})$ stabilizes $\| \ \|$?

Need $g_{ij} \in \mathcal{O}$ $\forall i,j$

if $i < j$ and $\|e_i\| > \|e_j\|$ then

need $g_{ij} \in \omega_F \mathcal{O}_F$

These are all conditions

Stabiliser $= \{ g \in GL_n(\mathcal{O}_F) \mid g \pmod{\omega_F} \in$ standard parabolic$\}$

These gps are parahorics subgps of GL_n.

In general, parahorics are stabilisers of facets, $\exists$ points with non parahoric stabilisers.

PGL: buildings norms on V / scalar mult.

$\| \ \|_1 \sim \| \ \|_2$ if $\exists c > 0$ st $\|v\|_1 = c\|v\|_2 \ \forall v$

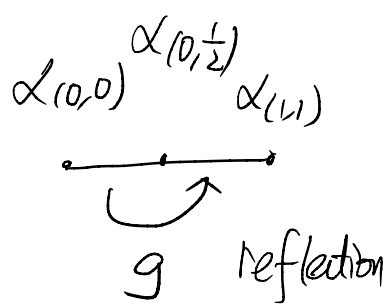
$n=2$

e_1, e_2 basis of V

$$\alpha_{(c_1,c_2)}(\lambda_1 e_1 + \lambda_2 e_2) = \min\{v(\lambda_1)+c_1, v(\lambda_2)+c_2\}$$

$$\alpha_{(c_1,c_2)} \sim \alpha_{(c_1+d, c_2+d)}$$

$$g = \begin{pmatrix} 0 & \varpi \\ 1 & 0 \end{pmatrix} \qquad g\alpha_{(0,0)} = \alpha_{(0,1)}$$



Building for PGL_2 is a $(q+1)$-valent tree

$g \notin$ any parahoric.

<u>Thm</u> Any two pts lie in a common apartment.

<u>Pf</u> $(GL(V))$ $\quad G = IwI \quad I =$ Iwahori

$W =$ affine Weyl

$\cong \mathbb{Z}^n \rtimes S_n$

Pick two points $\|\cdot\|_1$ $\|\cdot\|_2$

Pick two "good" bases $\{e_i\}$ $\{f_i\}$

so that max formula hold $\|e_1\| \geq \|e_2\|$

$\exists g$ s.t $g\{f_i\} \to \{e_i\}$

$g = \tau_1 w \tau_2$

rescales & permutes basis.

$\underbrace{\tau_2 \{f\}}_{\text{doesn't change } \|\cdot\|_2} = w^{-1} \tau_1^{-1} \{e_3\}$

change as basis.

other approach to GL_n building

(V/F f.d. v. space)

<u>Def</u> A filtration of V is a sequence

$\{L_r\}_{r \in \mathbb{R}}$ of $\mathcal{O}_F$-lattices in V s.t

- $L_r \subseteq L_s$ if $r > s$
- $L_{r+1} = \omega_F L_r$ $\forall r$.

$\dfrac{\mathcal{O}_F^n}{\omega \mathcal{O}_F^n}$ -finite

$\{\text{filtrations}\} = \{\text{norms}\}$

Given $\|\cdot\|$

$L_r = \{x \in V \mid \alpha(x) \geq r\}$

other way : $\alpha(v) = \sup_r \{r \mid v \in L_r\}$

$\mathcal{G}$ Chevalley gp scheme $/\mathbb{Z}$

$\cup\!\!\mid$

T torus Φ root sys.

$\alpha \in \Phi$ $\quad U_\alpha \cong \mathbb{G}_a$ $\quad k$ = local field

root subgp

$$k \xrightarrow{e_\alpha} U_\alpha(k)$$

Define $\varphi_\alpha : U_\alpha(k) \to \mathbb{R} \cup \{\infty\}$

$$\varphi_\alpha(u) = v(e_\alpha^{-1}(u))$$

More generally.
if $\lambda \in X_*(T) \otimes \mathbb{R}$

$$\varphi_\alpha(u) = v(e_\alpha^{-1}(u)) + \langle \lambda, \alpha \rangle$$

$\rightsquigarrow$ Filtrations of root gps U_α by cpt open subgps $U_{\alpha,r} := \varphi_\alpha^{-1}([\alpha, \infty])$

The functions $\{\varphi_\alpha\}_{\alpha \in \Phi}$ satisfy (valuations of root datum)

Points on the apartment in reduced building $\longleftrightarrow$ valuations of root data.

(V0) if $t \in T(K)$

$$\varphi_\alpha(tut^{-1}) = \varphi_\alpha(u) + v(\alpha(t))$$

(V1) $U_{\alpha,r}$ are open & bounded subgp

$U_{\alpha,\infty} = \{1\}$

(V2) Let M_α = non-identity coset of T in

$\langle T, S_\alpha \rangle \subseteq N_G(T)$

Let $m \in M_\alpha$ then the fn

$F(u) = \varphi_\alpha(u) - \varphi_\alpha(mum^{-1})$ is constant on

$U_\alpha \setminus \{1\}$.

(V3) If α, β linearly indep. roots

$$[U_{\alpha,r}, U_{\beta,s}] \subseteq \prod_{\substack{i,j \\ i\alpha+j\beta \in \Phi}} U_{i\alpha+j\beta,\, ir+js}$$

$$[e_\alpha(x), e_\beta(y)] = \prod_{i,j} e_{i\alpha+j\beta}(C_{ij}^{\alpha,\beta} x^i y^j) \quad C_{ij}^{\alpha\beta} \in \mathbb{Z}$$

$$|C_{ij}^{\alpha\beta}| < 4$$

(V4) $\varphi_{2\alpha} = 2\varphi_\alpha|_{U_{2\alpha}}$

(V5) if $u \in U_\alpha(K)$ & $u', u \in U_{-\alpha}(K)$

and $u'uu'' \in M_\alpha$ then $\varphi_{-\alpha}(u') = -\varphi_\alpha(u)$.

$$\begin{tikzcd}
\text{Apartment} \arrow[r, rightsquigarrow] & \text{torus} \\
\text{pt} \arrow[u, hook] \arrow[d] & \{ \arrow[u, hook] \\
\text{val of root datum} & U_\alpha \cdots
\end{tikzcd}$$

If $r \geq 0$ $\{\varphi_\alpha\}$ val of root datum

Define $G_{\varphi,r} = \langle U_{\alpha,r} \mid \alpha \in \Phi \rangle \leq G(K)$

(Moy - Prasad filtration)

sequence of open cpt subgps.

Serious theorem

$\exists$ smooth gp gp $\mathcal{G}/\mathcal{O}_F$

s.t $\mathcal{G}(F) = G(F)$ and $\mathcal{G}(\mathcal{O}_F) \cong G_{\varphi,r}$

$G_{\varphi,r}$ only depends on pt in building
(not which apt it's in.)

$\rightsquigarrow G_{x,r}$ $x \in \mathcal{B}$.

Serious thm

Let (π, V) be an irr. repn of $G(F)$

Then $\inf\{r \mid \exists\, x \in \mathcal{B} \mid V^{G_{x,r+}} \neq 0\} \in \mathbb{Q}$

and the infimum is achieved.

This is the depth.

Generality ?

Usually require : torus split under a tamely ramified ext

<u>Q</u>: What to do with anisotropic factors?

Where might tamely ramified assumption come up?

Tori: Let T be comm. gp scheme $/K$

Want: Neron model.

$\mathcal{T}$: smooth gp scheme $/\mathcal{O}_K$

[BLR] $\mathcal{T}(\mathcal{O}_K) = T(K)$

We want $\mathcal{T}(\mathcal{O}_K^{sh}) = \max'l$ bounded in $T(K^{ur})$.

strict Hesselization

Uniqueness?

<u>Lemma</u> Let $\begin{array}{c} \mathcal{X} \\ \downarrow \\ \operatorname{spec} \mathcal{O} \end{array}$ be smooth & affine

generic fiber X/K Then

$$\mathcal{O}_{\mathcal{X}} = \{ f \in \mathcal{O}_X \mid f(X(\mathcal{O}^{sh})) \subseteq \mathcal{O}^{sh} \}$$

Existence? $T = \mathbb{G}_m$, $\hat{T} = \mathbb{G}_m$.

Descent: T/K torus L/K Gal. ext.

$\mathcal{T}_L$ Neron model of T_L.

$\Gamma = \mathrm{Gal}(L/K)$

Want $\mathcal{T} = (\mathrm{Res}_{\mathcal{O}_L/\mathcal{O}_K} \mathcal{T}_L)^{\Gamma}$

easy smooth if $|\Gamma|$ invertible in $\mathcal{O}_K$