Representation of groups

Let G be a finite group.

A representation (V, π) of G is
a finite-dimensional $\mathbb{C}$-vector space V and
a homomorphism $\pi : G \to GL(V)$ ← general linear group of V

$g \mapsto \pi(g)$

= set of bijective linear transformations mapping $V \to V$

= set of automorphisms of V (bijective; $V \to V$)

linear transformation $\pi(g) : V \to V$, $v \mapsto \pi(g)v$

- $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ — π is a group homomorphism

  ie $\pi(g_1 g_2)v = \pi(g_1)[\pi(g_2)v]$ — V has a G-action

- $v \mapsto \pi(g)v$ is often written simply as $v \to gv$
- sometimes refer to the representation by π, or by V.
- dimension of a representation $\dim \pi = \dim(V)$

eg. Let $G = S_n$ (Symmetric group)

One possible representation of G is
$V = \mathbb{C}^n$ and
$\pi : g \in S_n \mapsto$ permutation of the coordinates

$\dim \pi = n$

eg. Let $G = C_n$ Cyclic group of order n
with generator γ.

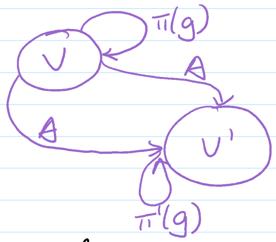
There are n obvious representations of G given by $V = \mathbb{C}$ and
$\gamma v = \zeta v$ where $\zeta \in \mathbb{C}$ is a nth root of unity.

$\dim \pi = 1$

Let $\pi : G \to GL(V)$ and $\pi' : G \to GL(V')$ be two representations of G.

We say the two representations V and V' are isomorphic $(V \simeq V')$
if there exists $A : V \to V'$ such that $A(\pi(g)v) = \pi'(g)A(v)$

↑ G-equivariant isomorphism from V to V'



ie $V \simeq V'$ if there exists a G-equivariant isomorphism from V to V'.

We write $V \cong V'$ if such an isomorphism has been fixed.

Let G be a finite group.
Let V be a representation of G and A a $\mathbb{C}$-vector space. $g : v \mapsto gv$ with $(g_2 g_1)v = g_2(g_1 v)$

Let G be a finite group.
Let V be a representation of G and A a $\mathbb{C}$-vector space. $g: v \mapsto gv$ with $(g_2 g_1)v = g_2(g_1 v)$

Then $V \otimes_{\mathbb{C}} A$ is a representation of G via

$$g\,(v \otimes a) = (gv) \otimes a$$

→ verify this is indeed an action

$$(g_1 g_2)(v \otimes a) = (g_1 g_2 v) \otimes a = (g_1(g_2 v)) \otimes a = g_1 (g_2 v \otimes a) = g_1 (g_2 (v \otimes a))$$

$\mathrm{Hom}_{\mathbb{C}}(A, V)$ set of homomorphism ($\mathbb{C}$-module) from A to V ie $\{\phi : A \to V\}$

$\mathbb{F}$-vector space / R-module: $f(x+y) = f(x) + f(y)$, $f(rx) = rf(x)$ — group homomorphism + scalar multiplication property

is also a representation of G via

$$(g\phi)(a) = g[\underbrace{\phi(a)}_{\in V}]$$

If $\dim_{\mathbb{C}} A = K$, then $V \otimes_{\mathbb{C}} A \simeq \underbrace{V \oplus \cdots \oplus V}_{K \text{ copies}} \simeq \mathrm{Hom}_{\mathbb{C}}(A, V)$

Let $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ ← dual space of V

V^* is a representation of G via

$$(g\phi)(v) = \phi(g^{-1} v)$$

$$(g_2 g_1 \phi)(v) = \phi((g_1^{-1} g_2^{-1})v) = \phi(g_1^{-1}(g_2^{-1} v)) = (g_1 \phi)(g_2^{-1} v) = (g_2(g_1 \phi))(v)$$

$\mathrm{Hom}_{\mathbb{C}}(V, A)$ is also a representation via

$$(g\phi)(v) = \phi(g^{-1} v)$$

If $\dim_{\mathbb{C}} A = K$, $\mathrm{Hom}_{\mathbb{C}}(V, A) \simeq \underbrace{V^* \oplus \cdots \oplus V^*}_{K \text{ copies}}$

Let V and V' be two representations of G.

We define $V \otimes_G V'$ as the quotient of $V \otimes_{\mathbb{C}} V'$ by the relation $gv \otimes v' = v \otimes gv'$.

$\mathrm{Hom}_G(V, V')$ as the set of G-equivariant linear transformations from V to V'.

are vector spaces but
are not representations of G (ie. no natural G-action)

Irreducible representations

A representation V of G is called irreducible if

it contains no proper sub-space which is invariant under the action of G

non-trivial, $\dim \geq 1$, $\neq \{0\}$, $\neq V$

If $W \subseteq V$ is invariant under the action of G, then if $w \in W, g \in G$ then $gw \in W$.

eg. For the symmetric group S_n,
$V = \mathbb{C}^n$ is a representation with $g \in S_n$ → permuting coordinates

Consider $W_1 = \{(x, \ldots, x) \mid x \in \mathbb{C}\}$,
$W_2 = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid x_1 + \cdots + x_n = 0\}$

W_1, W_2 are non-trivial subspaces of V that is invariant under G.

So V is not irreducible.

Notice that W_1, W_2 are two irreducible representation of G and $W_1 \oplus W_2 = V$.

Lemma Any representation of G is a direct sum of irreducible ones.

Proof

Let V be a representation of G with G-action $g \circ v$.

Goal: keep splitting V

Ideee: split V into W and $W^\perp$

Pick an inner product $\langle\ ,\ \rangle$ on V that is G-invariant.

Pick any positive definite $(\ ,\ )$. if $g \in G$, $\langle gv, gw\rangle = \langle v,w\rangle$

Define $\langle v, w\rangle = \sum_{g\in G} (gv, gw) \leftarrow (v,v) \geq 0$

If $h \in G$, then we want $\langle hv, hw\rangle = \langle v,w\rangle$.

$$\begin{aligned}
\langle hv, hw\rangle &= \sum_{g\in G} (hgv, hgw) \\
&= \sum_{hg\in G} (hgv, hgw) \\
&= \sum_{g\in G} (gv, gw)
\end{aligned}$$

If V is not irreducible, then there exists a non-trivial G-invariant subspace $W \subsetneq V$. $\langle v, w\rangle$

So if $w \in W$, $g \circ w \in W$ $\forall g \in G$

[$(g_2 g_1) \circ w = g_2 \circ (g_1 \circ w)$ $1 \circ w = w$ are inherited from V.]

So W still has a G-action $\circ: G \times W \to W$

So W is still a representation of V.

Let $W^\perp$ be the orthogonal complement of W.

So if $u \in W^\perp$, then $0 = \langle u, w\rangle$ $\forall w \in W$

$= \langle gu, gw\rangle$ $\forall g \in G, w \in W$

Since $\langle , \rangle$ is G-invariant

$\Rightarrow gu \in W^\perp$ $\forall g \in G$

So $W^\perp$ is also G-invariant.

So $W^\perp$ is still a representation of V.

So now we can write V as the direct sum of two representations

$V = W \oplus W^\perp$

Now we split W and $W^\perp$ using the same method.

Repeat this splitting process until each representation is irreducible.

(induction on the dimension)

Schur's Lemma

Let V and V' be two irreducible representations of G.
Then the $\mathbb{C}$-vector space $\mathrm{Hom}_G(V, V')$ is 0-dimensional if $V \not\simeq V'$ and
1-dimensional if $V \simeq V'$.

The space $\mathrm{Hom}_G(V, V)$ is canonically isomorphic to $\mathbb{C}$.