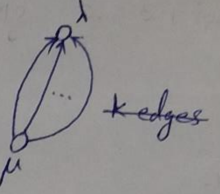
Recall:

• Branching graph: Vertices: $\bigcup_{n\geq 0} \hat{S_n}$ $\lambda \in \hat{S_n}$ V^λ corresponding irr. representation

Edges: $\lambda \in \hat{S_n}$, $\mu \in \hat{S_{n-1}}$

k edges if $k = \dim \mathrm{Hom}_{S_{n-1}}(V^\mu, V^\lambda)$



• Gelfand-Tsetlin basis (GZ-basis)

path $T = \lambda_0 \nearrow \lambda_1 \nearrow \cdots \nearrow \lambda_n = \lambda \rightsquigarrow V^\lambda = \bigoplus_T V_T \quad V_T = \langle v_T \rangle$

$\{v_T\}_T \subset V^\lambda$ GZ-basis

• $\mathrm{Spec}(n) := \left\{ \alpha(v_T) = (a_1, \ldots, a_n) \in \mathbb{C}^n \mid v_T \in \{v_T\}_T \subset V^\lambda, \lambda \in \hat{S_n}, \forall i \ a_i \text{ eigenvalue of YJM element } X_i \text{ on } v_T \right\}$

$\mathrm{Spec}(n) \xleftrightarrow{1-1}$ the set of paths in the first n levels of Branching graph.

$\alpha, \beta \in \mathrm{Spec}(n)$, $\alpha \sim \beta$ if v_α and v_β belong to the same irr. rep. of S_n.

Equivalently, T_α and T_β have the same end.

$$|\mathrm{Spec}(n)/\sim| = |\hat{S_n}|$$

Parallel story:

• Young graph $\mathbb{Y}$: Vertices: Young diagram □ □□ ⊟ ...

Edges: $\underset{\mu}{\square\square}$ — $\underset{\lambda}{⊞}$ if $\lambda / \mu = \square$

• $\mathrm{Cont}(n) = \{$content vector of a path in the Young graph$\}$

$\mathrm{Cont}(n) \xleftrightarrow{1-1} \{$paths in the first n levels of Young graph, $\mathbb{Y}_n\}$

- $\approx$ on $\mathrm{Cont}(n)$ s.t. $|\mathrm{Cont}(n)/\approx| = |$The partitions of $n|$

Goal:

Branching thm (Thm 5.8)

- ① Young graph $\mathbb{Y}$ = branching graph of S_n.
- ② Spec. of $GZ(n)$ = the space of paths in $\mathbb{Y}_n$.
- ③ $\mathrm{Spec}(n) = \mathrm{Cont}(n)$
- ④ $\sim \; = \; \approx$

$$(③ + ④ \Longrightarrow ① \;\&\; ②)$$

$$\begin{tikzcd}
\text{Branching graph} \arrow[r, equal] \arrow[d, leftrightarrow, "1-1"'] & \text{Young graph} \arrow[d, leftrightarrow, "1-1"] \\
\mathrm{Spec}(n) \arrow[r, equal, "③ \oplus ④"'] & \mathrm{Cont}(n) \\
\sim & \approx
\end{tikzcd}$$

Def. $\mathrm{Cont}(n)$ is the set of content vectors $\alpha = (a_1, \dots, a_n)$ s.t.

(1) $a_1 = 0$

(2) $\forall q > 1$ $\{a_q - 1, a_q + 1\} \cap \{a_1, \dots, a_{q-1}\} \neq \phi$

(3) If $a_p = a_q = a$ for some $p < q$, then $\{a-1, a+1\} \subset \{a_{p+1}, \dots, a_{q-1}\}$.

Clearly $\mathrm{Cont}(n) \subseteq \mathbb{Z}^n$.

Example. $n=1$ $\mathrm{Cont}(1) = \{\alpha = (a_1) = (0)\}$

$n=2$ $\mathrm{Cont}(2) = \{\alpha = (a_1, a_2) |$

By (1) $a_1 = 0$

By (2) $\{a_2 - 1, a_2 + 1\} \cap \{a_1\} \neq \phi \Longleftrightarrow \{a_2 - 1, a_2 + 1\} \cap \{0\} \neq \phi$

$\Longleftrightarrow a_2 - 1 = 0$ OR $a_2 + 1 = 0$

$\Longleftrightarrow a_2 = 1$ OR $a_2 = -1$

$\mathrm{Cont}(2) = \{(0,1), (0,-1)\}$

$n=3$ $\text{Cont}(3)=\{\alpha=(a_1,a_2,a_3) \mid a_1=0$

By (2) $q=2$ same as $\text{Cont}(2)$ | $q=3$ $\{a_3-1,a_3+1\}\cap\{a_1,a_2\}\neq\phi$

$a_2=1$ OR $a_2=-1$ | $\Longleftrightarrow \{a_3-1,a_3+1\}\cap\{0,a_2\}\neq\phi$

$a_2=1$	$a_3-1=0$	$a_3=1$	$\cancel{(0,1,1)}$	$a_2=-1$	$a_3-1=0$	$a_3=1$	$(0,-1,1)$
	$a_3-1=1$	$a_3=2$	$(0,1,2)$		$a_3-1=-1$	$a_3=0$	$\cancel{(0,-1,0)}$
	$a_3+1=0$	$a_3=-1$	$(0,1,-1)$		$a_3+1=0$	$a_3=-1$	$\cancel{(0,-1,-1)}$
	$a_3+1=1$	$a_3=0$	$\cancel{(0,1,0)}$		$a_3+1=-1$	$a_3=-2$	$(0,-1,-2)$

By (3) $\alpha=(0,\underset{a_p}{\underset{\shortparallel}{1}},\underset{a_q}{\underset{\shortparallel}{1}}) \Longrightarrow \{0,2\}\subset\phi$ Impossible

$\alpha=(\underset{a_p}{\underset{\shortparallel}{0}},1,\underset{a_q}{\underset{\shortparallel}{0}}) \Longrightarrow \{-1,1\}\subseteq\{1\}$ Impossible

$\text{Cont}(3)=\{(0,1,2),(0,1,-1),(0,-1,1),(0,-1,-2)\}$

$\text{Cont}(4)=\{(0,1,2,3),(0,1,2,-1),(0,1,-1,2),(0,1,-1,0),(0,1,-1,-2),(0,-1,1,2),$
$(0,-1,1,0),(0,-1,1,-2),(0,-1,-2,1),(0,-1,-2,-3)\}$

<u>Thm.1.</u> $\text{Spec}(n)\subset\text{Cont}(n)$

<u>Lemma 2.</u> In $\alpha=(a_1,\dots,a_n)$, if $a_i=a_{i+2}=a_{i+1}-1$ for some i, then $\alpha\notin\text{Spec}(n)$

<u>Proof.</u> Otherwise suppose that $\alpha\in\text{Spec}(n)$. By properties of $\text{Spec}(n)$

$s_iv_\alpha=v_\alpha$, $s_{i+1}v_\alpha=-v_\alpha \Longrightarrow s_is_{i+1}s_iv_\alpha=-v_\alpha$ but $s_{i+1}s_is_{i+1}v_\alpha=v_\alpha$ $\times$. $\square$

<u>Proof of Thm.1.</u> Let $\alpha=(a_1,\dots,a_n)\in\text{Spec}(n)$. $X_1=0$ so its eigenvalue is $0\Rightarrow a_1=0$

Con. (2) and (3) of definition can be verified by induction on n.

$n=2$ $X_2=(1\ 2)\in\mathbb{C}[S_2]$ Irr. rep. of S_2 are 1 and ε_2.

$1: S_2\longrightarrow GL(\mathbb{C})=\mathbb{C}^*$

$g\longmapsto 1$ $\quad \text{Eig}(X_2)=1$

$\varepsilon_2: S_2\longrightarrow GL(\mathbb{C})=\mathbb{C}^*$

$g\longmapsto \text{sgn}(g)$ $\quad \text{Eig}(X_2)=\text{sgn}(X_2)=-1$ ✓

$\therefore \text{Spec}(2)=\{(0,1),(0,-1)\}=\text{Cont}(2)$.

<u>Induction:</u> If $Spec(2) \subseteq Cont(2)$

Want $Spec(3) \subseteq Cont(3)$

Need to check: $Spec(3)$ satisfies ① ② ③.

① $a_1 = 0$ since $X_1 = 0$

② $Spec(3) = \{\alpha = (a_1, a_2, a_3)\}$ Assume otherwise,

$$\{a_3-1, a_3+1\} \cap \{a_1, a_2\} = \phi$$

$\Longrightarrow a_2 \neq a_3 \pm 1$ so by properties of Spec

$$\alpha' = S_2 \cdot \alpha = (a_1, a_3, a_2) \in Spec(3)$$

$$\Longrightarrow (a_1, a_3) \in Spec(2) \underset{\text{Ind. hyp.}}{\subset} Cont(2)$$

but $\{a_3+1, a_3-1\} \cap \{a_1\} = \phi$ $\dot{X}$.

③ If $a_p = a_q = a$ for some $p<q<n$ ✓

Only take $q=n$:

$$(\cdots, \overset{a}{\overset{\shortparallel}{a_p}}, \underbrace{}_{\text{largest possible}} \overset{\text{No } a\text{'s}}{\boxed{- - -}}, \overset{a}{\overset{\shortparallel}{a_n}})$$

If no such p exists, then ③ is true automatically

Assume that

$$\alpha = (a_1, \cdots, \overset{a}{\overset{\shortparallel}{a_p}}, \underset{\text{No } a-1}{\overset{\text{No } a\text{'s}}{\boxed{\cdots}}}, \overset{a}{\overset{\shortparallel}{a_n}})$$

Want to show $\{a-1, a+1\} \subseteq \{a_{p+1}, \cdots, a_{n-1}\}$. Assume otherwise,

~~then~~ $a+1$ occurs in $\{a_{p+1}, \cdots, a_{n-1}\}$ <u>at most once</u>.

If there are two, by Ind. hyp.

$$\{\underset{a}{\underset{\shortparallel}{a+1-1}}, a+1+1\} \subseteq \{a_{p+1}, \cdots, a_{n-1}\} \ \dot{X}.$$

So there are 2 cases:

① $(a_p, \cdots, a_n) = (a, \cdots, a)$

② $(a_p, \cdots, a_n) = (a, \cdots, a+1, \cdots, a)$

Case 1: switch $(\underbrace{\boxed{a, a,}}_{\text{less than } n} \cdots) \in \cancel{Spec}$ contradicting with ③ for lower n

Case 2: switch $(\cdots, \overline{\overline{a, a+1, a,}} \cdots) \notin Spec(n)$ $\dot{X}$ $\square$

Lemma 2

a_i a_{i+1} a_{i+2}

Verify ③ $Spec(7) \subseteq Cont(7)$

$$\alpha = (a_1, a_2, \underset{\underset{\substack{\nearrow \\ 3 \max.}}{a}}{\overset{\shortparallel}{a_3}}, \underbrace{a_4, a_5, a_6}_{\substack{\text{No } a\text{'s} \\ \text{No } a+1}}, \underset{a}{\overset{\shortparallel}{a_7}})$$

Want to show $\{a-1, a+1\} \subseteq \{a_4, a_5, a_6\}$

Assume $a-1 \notin \{a_4, a_5, a_6\}$

Note: $a+1$ occurs in $\{a_4, a_5, a_6\}$ at most once

$\Longrightarrow$ Case 1 $\underbrace{\{a_4, a_5, a_6\}}_{\text{No } a+1}$ OR Case 2 $\{a_4, a_5, a_6\}$ with $a_4 \neq a+1$, $a_5 = a+1$, $a_6 \neq a+1$

Otherwise, $\{a_4, a_5, a_6\}$ with $a_4 = a+1$, $a_6 = a+1$ $\Longrightarrow$ since $Spec(6) \subseteq Cont(6)$

$\{\underset{a}{a+1-1}, a+1+1\} \subseteq \{a_5\}$ $\dot{X}$

Case 1: switch

$(\cdots, \underbrace{\boxed{a, a,}}_{\text{less than } 7} \cdots)$

not satisfying ③ for $Spec(6)$

Case 2: switch

$(\cdots, \underbrace{\overline{\overline{a, a+1, a,}}}_{\text{by Lemma 2}} \cdots) \notin Spec(7)$

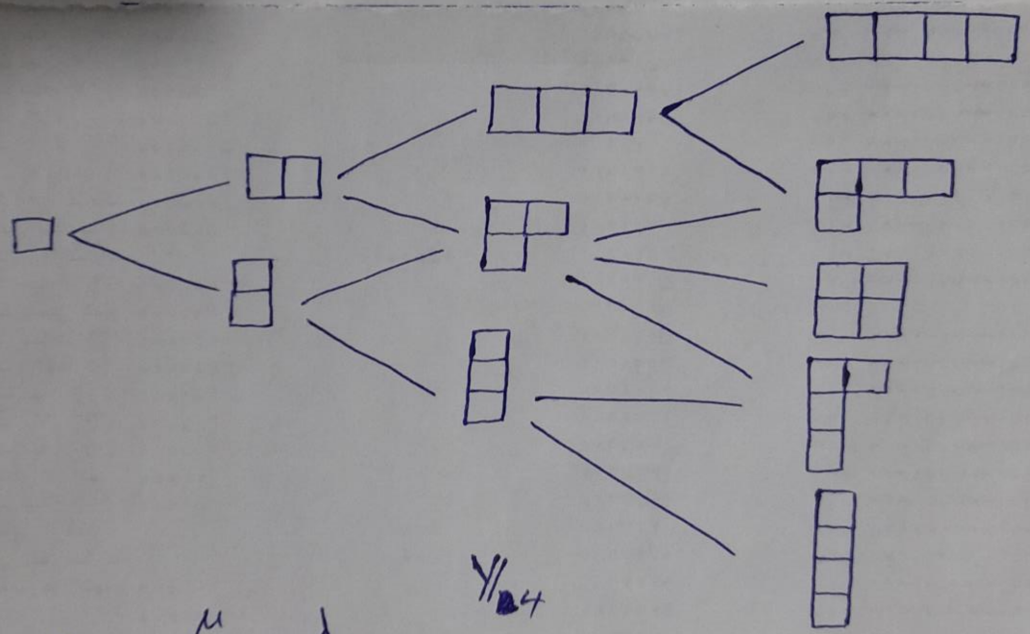
Remark. We could also assume

$$a+1 \notin \{a_{p+1}, \dots, a_{n-1}\}.$$

then $a-1$ occurs in $\{a_{p+1}, \dots, a_{n-1}\}$ at most once and

Case 2: switch $(\dots, \underline{\underline{a, a-1, a}}, \dots) \cdot \dot{X}$.

by another version of lemma 2



$\mathbb{Y}_{\leq 4}$

Def. Let μ and λ be two vertices in Young graph joined by a directed edge; i.e. $\mu \nearrow \lambda$. The content of $\lambda/\mu = \square$ is

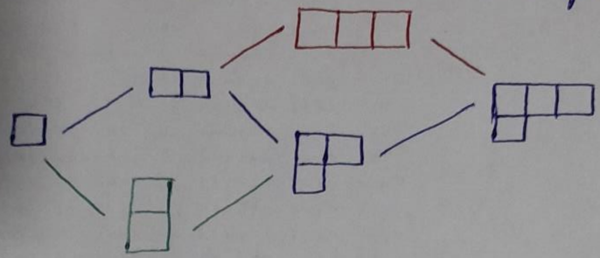
$$c(\lambda/\mu) = x\text{-coordinate of } \square - y\text{-coordinate of } \square$$

Convention. We use $\begin{array}{l} \ulcorner\!\longrightarrow x \\ \downarrow \\ y \end{array}$ as coordinate axes.

0	1	2	3	4	5
-1	0	1	2	3	4
-2	-1	0	1	2	3
-3	-2	-1	0	1	2
-4	-3	-2	-1	0	1
-5	-4	-3	-2	-1	0

Def. Young tableau of λ, $\mathrm{Tab}(\lambda)$, is the set of paths in $\mathbb{Y}$ from $\emptyset$ to λ.

Example



$$\mathrm{Tab}(n) := \bigcup_{|\lambda|=n} \mathrm{Tab}(\lambda)$$

the number of boxes

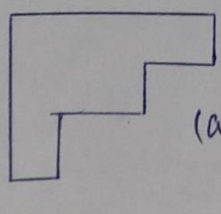
Now we want to show that $\mathrm{Cont}(n)$ is the set of content vectors of Young tableaux.

Prop. 3. There is a bijection $\mathrm{Tab}(n) \longrightarrow \mathrm{Cont}(n)$ which maps a tableau $T = \lambda_0 \nearrow \dots \nearrow \lambda_n$ to the vector $(c(\lambda_1/\lambda_0), \dots, c(\lambda_n/\lambda_{n-1}))$.

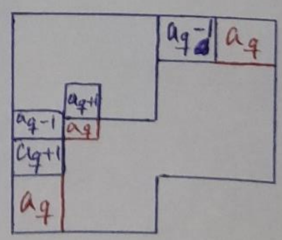
Proof. Check con. ①, ②, ③ for content vectors:

① $\square$ starting box with 0 content.

②



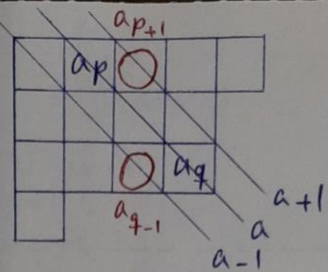
$(a_1, \dots, a_{q-1})$



3 possible boxes for a_q

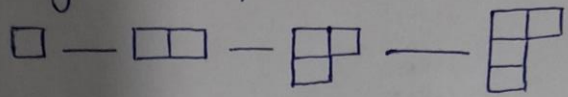
Original diagram has a box with content $a_q - 1$ or $a_q + 1$.

③ $a_p = a_q = a$



$\{a+1, a-1\} \subseteq \{a_{p+1}, \dots, a_{q-1}\}$

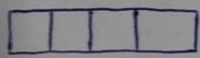
($\Longleftarrow$) Given a content vector. We can construct a Young diagram box-by-box. This process determines the paths in $\mathbb{Y}$ uniquely. $\square$

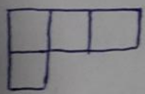
Example.  $(0,1,-1,-2)$

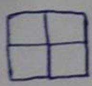
Def. $\alpha = (a_1, \dots, a_n) \in \mathbb{C}^n$, $s_i \in S_n$ is admissible with respect to α if $a_{i+1} \neq a_i \pm 1$. $\quad s_i \cdot \alpha = (\dots, a_{i+1}, a_i, \dots)$

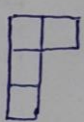
Def. $\alpha, \beta \in \mathrm{Cont}(n)$, $\alpha \approx \beta$ if β is an admissible transposition of the entries of α.

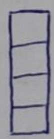
Example. S_4

 $(0,1,2,3)$ dimension 1

 $(0,1,2,-1)$, $(0,1,-1,2)$, $(0,-1,1,2)$ dimension 3

 $(0,1,-1,0)$, $(0,-1,1,0)$ dimension 2

 $(0,1,-1,-2)$, $(0,-1,1,-2)$, $(0,-1,-2,1)$ dimension 3

 $(0,-1,-2,-3)$ dimension 1

Paths $T = \lambda_0 \uparrow \dots \uparrow \lambda_n$ in $\mathrm{Tab}(n)$, $T \sim U$ if $\lambda_n = \eta_n = \lambda$.
$U = \eta_0 \uparrow \dots \uparrow \eta_n$

Want to show $\mathrm{Cont}(n)/\approx \; = \mathrm{Tab}(n)/\sim$.

Lemma 4. Any two Young tableau $T_1, T_2 \in \mathrm{Tab}(\lambda)$ can be obtained from each other by a sequence of admissible transpositions.

Proof. $T \in \mathrm{Tab}(\lambda)$, $\lambda = (\lambda_1, \dots, \lambda_k)$ any Young tableau

$$\alpha(T) \xrightarrow{\text{transform}} \alpha(T^\lambda)$$

where T^λ is the following tableau:

$$
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & \cdots & \lambda_1 \\
\hline
\lambda_1+1 & \cdots & \lambda_1+\lambda_2 & \\
\hline
\vdots & & & \\
\hline
n-\lambda_k+1 & \cdots & n & \\
\hline
\end{array}
$$

$$\alpha(T^\lambda)=(0,1,2,\dots,\lambda_1-1,-1,0,\dots,\lambda_2-2,-2,-1,\dots)$$

$$
\begin{array}{|c|c|c|}
\hline
1 & 3 & 4 \\
\hline
2 & 6 & 8 \\
\hline
5 \\
\cline{1-1}
7 \\
\cline{1-1}
\end{array}
\quad (0,-1,1,2,-2,0,\underline{-3},\underline{1})
\qquad
\begin{array}{|c|c|c|}
\hline
1 & 3 & 4 \\
\hline
2 & 6 & 7 \\
\hline
5 \\
\cline{1-1}
8 \\
\cline{1-1}
\end{array}
\quad (0,-1,1,2,\underline{-2},\underline{0},1,-3)
$$

$$
\begin{array}{|c|c|c|}
\hline
1 & 3 & 4 \\
\hline
2 & 5 & 7 \\
\hline
6 \\
\cline{1-1}
8 \\
\cline{1-1}
\end{array}
\quad (0,-1,1,2,0,\underline{-2},\underline{1},-3)
\qquad
\begin{array}{|c|c|c|}
\hline
1 & 3 & 4 \\
\hline
2 & 5 & 6 \\
\hline
7 \\
\cline{1-1}
8 \\
\cline{1-1}
\end{array}
\quad (0,\underline{-1},\underline{1},2,0,1,-2,-3)
$$

$$
\begin{array}{|c|c|c|}
\hline
1 & 2 & 4 \\
\hline
3 & 5 & 6 \\
\hline
7 \\
\cline{1-1}
8 \\
\cline{1-1}
\end{array}
\quad (0,1,\underline{-1},\underline{2},0,1,-2,-3)
\qquad
\begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
4 & 5 & 6 \\
\hline
7 \\
\cline{1-1}
8 \\
\cline{1-1}
\end{array}
\quad (0,1,2,-1,0,1,-2,-3)
$$

In each step trasposiitng i and $i+1$ is admissible because $c(i)\neq c(i+1)\pm 1$. Otherwise they are in the correct order. Therefore, all these actions are admissible when translated into $Cont(n)$. $\square$

Corollary 5. If $\alpha\in Spec(n)$ and $\alpha\approx\beta$, $\beta\in Cont(n)$, then $\beta\in Spec(n)$ and $\alpha\sim\beta$.

$$
\begin{array}{ccc}
Tab(n)/\sim & \xleftrightarrow{\;1-1\;} & Cont(n)/\approx \\
\rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\
Spec(n) & & Spec(n)
\end{array}
$$

Remark. Our chain of transpositions which connects T and T^λ in lemma 4 is minimal in the following sense.

$$s \in S_n, \quad s.T = T^\lambda$$

$$\ell(s) := \#\{(i,j) \in \{1,\dots,n\} \mid i<j,\ s(i) > s(j)\}$$

$$= \text{the number of inversions in } s.$$

s can be written as the product of $\ell(s)$ admissible transpositions s_i but not as a shorter product.

Example. In lemma 4

$$s = (3\ 4)(2\ 3)(6\ 7)(5\ 6)(7\ 8)$$

In fact, $\mathrm{Cont}(n)$ is a "totally geodesic" subset of $\mathbb{Z}^n$ for the action of S_n.

---

the partitions of $n \xleftrightarrow{1-1}$ conjugacy classes of S_n ③

① $\overset{1-1}{\longleftrightarrow}$ cycle types in S_n $\overset{1-1}{\longleftrightarrow}$ ②

$$① + ② \Longrightarrow ③$$

Def. $\sigma \in S_n$ is of cycle type $(k_1, k_2, \dots, k_\ell)$ if

① $\sigma = \alpha_1 \alpha_2 \cdots \alpha_\ell$ where α_i is a k_i-cycle

② α_i's are disjoint

③ $k_1 \geq k_2 \geq \dots \geq k_\ell$

④ 1's are included in $(k_1, \dots, k_\ell)$ for fixed points.

Example. $\sigma \in S_{10}$, $\sigma = (1\ 3\ 4)(5\ 10\ 8\ 7)$

σ has cycle type $(4,3,1,1,1)$.

① ✓ because $\sum_{i=1}^{\ell} k_i = n$.

Thm. $\sigma, \rho \in S_n$

σ and ρ are conjugate $\Longleftrightarrow$ they have the same cycle type.

Branching thm.

① Young graph $\mathbb{Y}$ = branching graph of S_n.

② Spec. of $GZ(n)$ = the space of paths in $\mathbb{Y}_n$.

③ $$\mathrm{Spec}(n) = \mathrm{Cont}(n)$$

④ $$\sim \; = \; \approx$$

Proof. $\mathrm{Cont}(n)/\approx \; = \mathrm{Tab}(n)/\sim \;\Longrightarrow |\mathrm{Cont}(n)/\approx| = p(n)$

$p(n)$ = the number of partitions of n

= the number of diagrams with n boxes.

By corollary 5, each equivalence class in $\mathrm{Cont}(n)/\approx$ either does not contain elements of the set $\mathrm{Spec}(n)$, or is a subset of some class in $\mathrm{Spec}(n)/\sim$. But

$$|\mathrm{Spec}(n)/\sim| = |\hat{S_n}| = p(n)$$

Therefore, each class of $\mathrm{Cont}(n)/\approx$ coincides with one of the classes of $\mathrm{Spec}(n)/\sim$.

$$\mathrm{Spec}(n) \subset \mathrm{Cont}(n) \Longrightarrow \mathrm{Spec}(n) = \mathrm{Cont}(n) \text{ and } \sim \, = \, \approx$$

$$\mathrm{Tab}(n)/\sim \;\xleftrightarrow{1-1}\; \mathrm{Cont}(n)/\approx$$

$$\quad\quad\quad\quad\quad\quad\quad\quad \| \quad\quad \|$$

$$\quad\quad\quad\quad\quad\quad\quad\quad \mathrm{Spec}(n)/\sim$$

$$\Longrightarrow \mathrm{Tab}(n) = \mathrm{Spec}(n) \text{ and } \sim \, = \, \sim \quad ② \checkmark$$

Hence, Young graph is the branching graph of the symmetric groups. □