Recall:

Prop. For any vector v_T, $T=\lambda_0\nearrow\ldots\nearrow\lambda_n$, $\lambda_i\in S_i^{\wedge}$ and any $k=1,\ldots,n-1$, $s_k\cdot v_T=\sum c_{T'}v_{T'}$, $c_{T'}\in\mathbb{C}$

$$T'=\lambda_0'\nearrow\ldots\nearrow\lambda_n',\ \lambda_i'\in S_i^{\wedge}\quad \text{s.t. } \lambda_i'=\lambda_i,\ i\neq k.$$

The action of s_k affects only the kth level of the branching graph.

- Affine Hecke algebra $H(2)$

$$H(2)=\langle Y_1,Y_2,s\rangle / s^2=1,\ Y_1Y_2=Y_2Y_1,\ sY_1+1=Y_2s.$$

Relations $\Longrightarrow$ Irr. repn. of $H(2)$ are one-dimensional or two-dimensional.

Example. $H(2)=\langle X_{n-1},X_n,s_{n-1}\rangle$ Relations ✓

✱ Relations between Coxeter gen. & YJM elements are stable under shifts of indices.

- Irr. repn. of $H(2)$, V

$\exists v\in V$, $Y_1v=av$, $Y_2v=bv$, $a,b\in\mathbb{C}$

v & sv linearly independent $\Longrightarrow Y_1=\begin{pmatrix}a&-1\\0&b\end{pmatrix}$, $Y_2=\begin{pmatrix}b&1\\0&a\end{pmatrix}$

$$s=\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$a\neq b\Longrightarrow Y_1=\begin{pmatrix}a&0\\0&b\end{pmatrix}$, $Y_2=\begin{pmatrix}b&0\\0&a\end{pmatrix}$

$$s=\begin{pmatrix}\frac{1}{b-a}&1-\frac{1}{(b-a)^2}\\1&\frac{1}{a-b}\end{pmatrix}$$

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Goal:

<u>Prop.1</u> ① Irr. repn. of S_n are defined over $\mathbb{Q}$.

② $\exists \{v_T\} \subset V^\lambda$ basis s.t. if $T' = s_i . T$

$$s_i . v_T = v_{T'} + \frac{1}{a_{i+1} - a_i} v_T$$

$$s_i . v_{T'} = \left(1 - \frac{1}{(a_{i+1}-a_i)^2}\right) v_T - \frac{1}{a_{i+1}-a_i} v_{T'}$$

• $\{v_T\}$ Young's seminormal form of V^λ

$$\Big\downarrow \text{normalize}$$

Young's orthogonal form of V^λ

<u>Prop.2</u> $\exists$ Young's orthogonal form of V^λ, $\{v_T\}$ s.t.

$$s_i = \begin{pmatrix} r^{-1} & \sqrt{1-r^{-2}} \\ \sqrt{1-r^{-2}} & -r^{-1} \end{pmatrix} \qquad r = a_{i+1} - a_i$$

Recall T^λ, Young tableau of $\lambda \in \hat{S_n}$ ordered horizontally

$T \in \mathrm{Tab}(\lambda)$, ~~s per~~ $s \in S_n$ $s . T^\lambda = T$,

$$\ell(T) := \ell(s)$$

Base case: $T' = s_i . T$ $\ell(T') > \ell(T)$

$$\alpha(T) = (a_1, \dots, a_n) \in \mathrm{Cont}(n)$$

for $b = a_{i+1}$, $a = a_i$ in $s = \begin{pmatrix} \frac{1}{b-a} & 1 - \frac{1}{(b-a)^2} \\ 1 & \frac{1}{a-b} \end{pmatrix}$

$$\Longrightarrow s_i . v_T = v_{T'} + \frac{1}{a_{i+1}-a_i} v_T$$

$$s_i . v_{T'} = \left(1 - \frac{1}{(a_{i+1}-a_i)^2}\right) v_T - \frac{1}{a_{i+1}-a_i} v_{T'}$$

Proposition 1. ✓

Corollary. Let $s \in S_n$, $sT^\lambda = T$,

$$s.v_{T^\lambda} = v_T + \sum_{R \in Tab(\lambda),\, \ell(R) < \ell(T)} \gamma_R v_R$$

γ_R rational numbers.

- i & $i+1$ same row $\Longrightarrow$ the tableau T unchanged

  $\begin{array}{|c|c|}\hline i & i+1 \\ \hline\end{array}$ $\quad a_{i+1} = a_i + 1 \Longrightarrow s_i . v_T = v_T$

- i & $i+1$ same column $\Longrightarrow$ s_i multiplies T by -1

  $\begin{array}{|c|}\hline i \\ \hline i+1 \\ \hline\end{array}$ $\quad a_{i+1} = a_i - 1 \Longrightarrow s_i . v_T = -v_T$

- i & $i+1$ distinct rows and colums: $a_{i+1} \neq a_i \pm 1$

T': i & $i+1$ swapped $\Longrightarrow s_i \curvearrowright V = \langle v_T, v_{T'} \rangle$

$$s_i = \begin{pmatrix} r^{-1} & \sqrt{1 - r^{-2}} \\ \sqrt{1 - r^{-2}} & -r^{-1} \end{pmatrix}$$

Proposition 2. ✓