ADAMS-BARBASCH-VOGAN EXPLICITLY

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Aim of today: Make the bijection in the ABV correspondence completely explicit.
Plan:

(1) Recollection of the statement (geometric version)
(2) Combinatorics
(3) Examples

1. Recollection of the statement

Fix G and $\check{G}$ dual (pinned) reductive groups. "Pinned" means that G and $\check{G}$ are equipped with Cartan and Borel subgroups $H \subset B \subset G$ and $\check{H} \subset \check{B} \subset \check{G}$, and bases for all the simple root spaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ and $\check{\mathfrak{g}}_\alpha \subset \check{G}$. I will adopt the convention that roots in B and $\check{B}$ are *negative*. (I think this is the opposite convention to Qixian.) "Dual" means that we have isomorphisms

$$\mathbb{X}^*(H) \cong \mathbb{X}_*(\check{H}) \quad \text{and hence} \quad \mathbb{X}_*(H) \cong \mathbb{X}^*(\check{H})$$

such that the sets Φ and Φ_+ of (positive) roots for G are identified with the sets of (positive) coroots for $\check{G}$, the sets $\check{\Phi}$ and $\check{\Phi}_+$ of (positive) coroots for G are identified with the sets of (positive) roots for $\check{G}$, and the $\check{(\cdot)}$ bijections are the same for G and $\check{G}$. Here $\mathbb{X}_* = \mathrm{Hom}(\mathbb{C}^\times, -)$ and $\mathbb{X}^* = \mathrm{Hom}(-, \mathbb{C}^\times)$.

I will also fix throughout an involution δ of the root datum of G, which lifts to a unique involution of G preserving the pinning, and write $\check{\delta} = -w_0\delta$ for the dual involution on $\check{G}$. The corresponding extended groups are $G^\Gamma = G \rtimes \{1, \delta\}$ and $\check{G}^\Gamma = \check{G} \rtimes \{1, \check{\delta}\}$.

1.1. Automorphic side.

Define a space

$$X = \{x \in G^\Gamma - G \mid x^2 \in Z(G) \text{ has finite order}\} \times \mathcal{B}$$

where $\mathcal{B} = G/B$ is the flag variety. The H-bundle $\tilde{\mathcal{B}} = G/N \to \mathcal{B}$ pulls back to an H-bundle $\pi\colon \tilde{X} \to X$. For $\lambda \in \mathfrak{h}^*$, define

$$\mathcal{D}_{X,\lambda} = \pi_*(\mathcal{D}_{\tilde{X}})^H \otimes_{S(\mathfrak{h})} \mathbb{C}_{\lambda-\rho}.$$

The automorphic side of the correspondence is the set

$$\mathfrak{X}_{\mathrm{Aut}}(\lambda) = \left\{\begin{matrix}\text{irreducible objects in} \\ \mathrm{HC}(\mathcal{D}_{X,\lambda}, G)\end{matrix}\right\} \cong \left\{ (Q, \gamma) \;\middle|\; \begin{matrix} Q \subset X \text{ is a } G\text{-orbit} \\ \gamma \text{ is an irreducible equivariant} \\ \text{twisted local system on } Q \end{matrix} \right\}.$$

The bijection from right to left is given by intermediate extension.

Relation to version in [ABV]: Note that we can write

$$\mathrm{HC}(\mathcal{D}_{X,\lambda}, G) = \bigoplus_x \mathrm{HC}(\mathcal{D}_{\mathcal{B},\lambda}, K_x),$$

where the sum is over conjugacy classes of strong involutions x and $K_x = Z_G(x) = G^{\mathrm{Ad}_x}$. If λ is integrally dominant, then we have an exact functor

$$\Gamma\colon \mathrm{HC}(\mathcal{D}_{\mathcal{B},\lambda}, K_x) \to \mathrm{HC}(\mathfrak{g}, K_x)_\lambda$$

which realises $\mathrm{HC}(\mathfrak{g}, K_x)_\lambda$ as a Serre quotient of the source. If λ is regular, then the functor is an equivalence.

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1.2. **Galois side.** Keep $\lambda \in \mathfrak{h}^*$ fixed. On the dual side, define another space

$$\check{X}_\lambda = \{y \in \check{G}^\Gamma - \check{G} \mid y^2 = \exp(2\pi i\lambda)\} \times \check{\mathcal{B}}_\lambda,$$

where $\check{\mathcal{B}}_\lambda = \check{G}_\lambda/\check{B}_\lambda$, for $\check{G}_\lambda = Z_{\check{G}}(\exp(2\pi i\lambda))$ and $\check{B}_\lambda = \check{B} \cap \check{G}_\lambda$. We also let

$$\check{G}^{alg} = \text{pro-algebraic universal cover of } \check{G}$$

and

$$\check{G}^{alg}_\lambda = \check{G}_\lambda \times_{\check{G}} \check{G}^{alg}.$$

The Galois side of the correspondence is

$$\mathfrak{X}_{\mathrm{Gal}}(\lambda) = \left\{\begin{matrix}\text{irreducible objects in} \\ \mathrm{HC}(\mathcal{D}_{\check{X}_\lambda}, \check{G}^{alg}_\lambda)\end{matrix}\right\} \cong \left\{ (\check{Q}, \check{\gamma}) \;\middle|\; \begin{matrix} \check{Q} \subset \check{X}_\lambda \text{ is a } \check{G}^{alg}\text{-orbit} \\ \check{\gamma} \text{ is an irreducible} \\ \text{equivariant local system on } \check{Q} \end{matrix} \right\}.$$

Note that we can write

$$\mathrm{HC}(\mathcal{D}_{\check{X}_\lambda}, \check{G}^{alg}) = \bigoplus_y \mathrm{HC}(\mathcal{D}_{\check{\mathcal{B}}_\lambda}, \check{K}^{alg}_\lambda),$$

where the sum is over conjugacy classes of elements y and

$$\check{K}^{alg}_\lambda = \check{G}^{\mathrm{Ad}_y}_\lambda \times_{\check{G}} \check{G}^{alg}.$$

Relation to version in [ABV]**:** If λ is integrally dominant, then we have a parabolic subgroup $\check{P}_\lambda \subset \check{G}_\lambda$ containing $\check{B}_\lambda$ with roots

$$\{\check{\alpha} \in \check{\Phi} \mid \langle \lambda, \check{\alpha}\rangle \leq 0\}.$$

If we set

$$\check{X}^{par}_\lambda = \{y \in \check{G}^\Gamma - \check{G} \mid y^2 = \exp(2\pi i\lambda)\} \times \check{G}_\lambda/\check{P}_\lambda,$$

then we have a full abelian subcategory

$$\mathrm{HC}(\mathcal{D}_{\check{X}^{par}_\lambda}, \check{G}^{alg}_\lambda) \to \mathrm{HC}(\mathcal{D}_{\check{X}_\lambda}, \check{G}^{alg}_\lambda)$$

given by pullback.

1.3. **The main theorem.**

Theorem 1.1 (cf., [ABV, Theorems 1.18 and 1.24])**.** *We have the following.*

(1) *There is a bijection*

$$\mathrm{LLC}\colon \mathfrak{X}_{\mathrm{Gal}}(\lambda) \xrightarrow{\sim} \mathfrak{X}_{\mathrm{Aut}}(\lambda).$$

(2) *The two perfect pairings*

$$K(\mathrm{HC}(\mathcal{D}_{X,\lambda}, G)) \otimes K(\mathrm{HC}(\mathcal{D}_{\check{X}_\lambda}, \check{G}^{alg})) \to \mathbb{Z}$$

given by

$$\langle [j_!\gamma], [j_!\check{\gamma}]\rangle = (-1)^{\ell(\check{\gamma})} e(\check{\gamma}) \delta_{\gamma, \mathrm{LLC}(\check{\gamma})}$$

and

$$\langle [j_{!*}\gamma], j_{!*}\check{\gamma}]\rangle = (-1)^{\ell(\check{\gamma})} e(\check{\gamma}) \delta_{\gamma, \mathrm{LLC}(\check{\gamma})}$$

coincide. Here j denotes the inclusion of an orbit, $\ell(\check{\gamma}) = \ell(\check{Q}, \check{\gamma}) = \dim \check{Q}$ and e is Kottwitz's sign.

(3) *If λ is integrally dominant, then the subgroup*

$$K(\mathrm{HC}(\mathcal{D}_{\check{X}^{par}_\lambda}, \check{G}^{alg}_\lambda)) \subset K(\mathrm{HC}(\mathcal{D}_{\check{X}_\lambda}, \check{G}^{alg}_\lambda))$$

is the orthogonal complement to the kernel of the quotient map

$$\Gamma\colon K(\mathrm{HC}(\mathcal{D}_{X,\lambda}, G)) \to \bigoplus_x K(\mathrm{HC}(\mathfrak{g}, K_x)_\lambda).$$

2. Combinatorics

2.1. Automorphic side.
We have

$$X/G \cong \{\text{strong involutions}\}/B.$$

By a theorem of Matsuki, for each strong involution x the Borel subgroup B contains a $\theta_x := \mathrm{Ad}_x$-stable Cartan subgroup. So each strong involution is conjugate by B to one normalising our chosen Cartan $H \subset B$. If x normalises H, then we have

$$\mathrm{Stab}_B(x) = B \cap G^{\theta_x} = (N \cap G^{\theta_x})H^{\theta_x},$$

where $N \subset B$ is the unipotent radical of B. An easy exercise with the definitions shows that the equivariant λ-twisted $\mathcal{D}$-modules on $Q = G \cdot x$ are identified with Harish-Chandra modules for the pair $(\mathfrak{h}, \mathrm{Stab}_B(x))$ on which $\mathfrak{h}$ acts by $\lambda + \rho$. Since $N \cap G^{\theta_x}$ is unipotent, it acts trivially on any irreducible module, so we obtain the following.

Proposition 2.1. *The set $\mathfrak{X}_{\mathrm{Aut}}(\lambda)$ is in natural bijection with*

$$\left\{ (x, \Lambda) \;\middle|\; \begin{array}{c} x \in N_G(H)\delta/H \text{ such that } x^2 \in Z(G) \text{ has finite order} \\ \Lambda \in \mathbb{X}^*(H^{\theta_x}) \text{ such that } d\Lambda = (\lambda + \rho)|_{\mathfrak{h}^{\theta_x}} \end{array} \right\}.$$

Here H acts on $N_G(H)\delta$ by conjugation.

Recall the structure of the group $N_G(H)$. We have the exact sequence

$$1 \to H \to N_G(H) \to W \to 1.$$

For each simple root α, we have a canonical lift

$$\tilde{s}_\alpha = \phi_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in N_G(H)$$

of the simple reflection $s_\alpha \in W$, where $\phi_\alpha \colon \mathrm{SL}_2 \to G$ is the root homomorphism determined by our chosen pinning.

Lemma 2.2. *We have the following.*

1. *The group $N_G(H)$ is generated by H and $\tilde{s}_\alpha$ for α simple, subject to the relations*
$$\tilde{s}_\alpha^2 = \check{\alpha}(-1) \quad \text{and} \quad \tilde{s}_\alpha \tilde{s}_\beta \cdots = \tilde{s}_\beta \tilde{s}_\alpha \cdots$$
for simple roots α and β, where there are $m_{\alpha,\beta}$ factors on both sides of the second relation.
2. *If $w \in W$, define $\tilde{w} = \tilde{s}_{\alpha_1}\tilde{s}_{\alpha_2}\cdots\tilde{s}_{\alpha_n}$, where $w = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_n}$ is a reduced word for w. (This is independent of the choice of reduced word by the braid relations.) Then*
$$N_G(H) = \coprod_{w \in W} H\tilde{w}.$$
3. *If $ww' = 1$ in W, then*
$$\tilde{w}\tilde{w}' = (\check{\rho} - w\check{\rho})(-1),$$
where $\check{\rho}$ is half the sum of positive coroots.

Using this lemma, we can write the set $\mathfrak{X}_{\mathrm{Aut}}(\lambda)$ as follows.

Proposition 2.3. *We have a natural parametrisation*

$$\mathfrak{X}_{\mathrm{Aut}}(\lambda) = \{(x = \exp(2\pi i h)\tilde{w}\delta, \Lambda)\}$$

where

$$w \in W, \quad h \in \frac{\mathfrak{h}_{\mathbb{Q}}}{\mathbb{X}_*(H) + (1 - w\delta)\mathfrak{h}_{\mathbb{Q}}}, \quad \text{and} \quad \Lambda \in \frac{\mathbb{X}^*(H)}{(1 - w\delta)\mathbb{X}^*(H)}$$

satisfy the conditions

$$w\delta(w) = 1 \tag{2.1}$$

$$(1+w\delta)\left(h+\frac{1}{2}\check{\rho}\right) \in \frac{\check{P}}{(1+w\delta)\mathbb{X}_*(H)} \tag{2.2}$$

and

$$\lambda+\rho-\Lambda \in \frac{(\mathfrak{h}^*)^{-w\delta}}{(1-w\delta)\mathbb{X}^*(H)} \tag{2.3}$$

Here

$$\mathfrak{h}_{\mathbb{Q}} := \mathbb{X}_*(H)\otimes\mathbb{Q}$$

and

$$\check{P} = \{h \in \mathfrak{h}_{\mathbb{Q}} \mid \langle\alpha, h\rangle \in \mathbb{Z} \textit{ for all } \alpha \in \Phi\}.$$

Proof. Exercise. Minor remarks:

- (2.1) and (2.2) are equivalent to $x^2 \in Z(G)$.
- $h \in \mathfrak{h}_{\mathbb{Q}}$ is equivalent to x^2 has finite order.
- (2.3) is equivalent to $d\Lambda = (\lambda+\rho)|_{\mathfrak{h}^{\theta_x}}$.

□

2.2. **Galois side.** Similarly, we have

Proposition 2.4. *The set $\mathfrak{X}_{\mathrm{Gal}}(\lambda)$ is in natural bijection with*

$$\left\{(y,\check{\Lambda}) \;\middle|\; \begin{matrix} y \in N_{\check{G}}(\check{H})\check{\delta}/\check{H} \textit{ such that } y^2 = e^{2\pi i\lambda} \\ \check{\Lambda} \in \mathbb{X}^*(\pi_0(\check{H}^{\theta_y})^{alg}) \end{matrix}\right\},$$

where $\theta_y = \mathrm{Ad}_y$ and

$$(\check{H}^{\theta_y})^{alg} = \check{H}^{\theta_y} \times_{\check{G}} \check{G}^{alg}.$$

To write this combinatorially, it will be convenient to use the element $-\delta = w_0\check{\delta} \in N_{\check{G}}(\check{H})\check{\delta}$ as a base point. Note that

$$(-\delta)^2 = 2\rho(-1) \in \check{H}.$$

Proposition 2.5. *We have a natural parametrisation*

$$\mathfrak{X}_{\mathrm{Gal}}(\lambda) = \{(y = e^{2\pi i h'}\tilde{w}(-\delta), \check{\Lambda})\}$$

where

$$w \in W, \quad h' \in \frac{\mathfrak{h}^*}{\mathbb{X}^*(H)+(1+w\delta)\mathfrak{h}^*}, \quad \textit{and} \quad \check{\Lambda} \in \frac{\check{P}^{w\delta}}{(1+w\delta)\mathbb{X}_*(H)}$$

satisfy the conditions

$$w\delta(w) = 1, \tag{2.4}$$

and

$$\lambda + \frac{1}{2}(1+w\delta)\rho - (1-w\delta)h' \in \frac{\mathbb{X}^*(H)}{(1-w\delta)\mathbb{X}^*(H)}. \tag{2.5}$$

Proof. Exercise. Minor remarks:

- I have identified, e.g., $\check{\mathfrak{h}}$ with $\mathfrak{h}^*$ etc.
- (2.4) and (2.5) are equivalent to $y^2 = \exp(2\pi i\lambda)$.
- $\check{P}$ is the character group of $\check{H}^{alg} = \check{H} \times_{\check{G}} \check{G}^{alg}$.

□

2.3. **A bijection.** Here is a bijection between the two sides. I am moderately confident it is the correct one.

Proposition 2.6. *The map*

$$(y = \exp(2\pi i h')\tilde{w}(-\delta), \check{\Lambda}) \mapsto (x = \exp(2\pi i h)\tilde{w}\delta, \Lambda)$$

given by

$$h = \frac{1}{2}(\check{\Lambda} - \frac{1}{2}(1 + w\delta)\check{\rho}) \quad \textit{and} \quad \Lambda = \lambda + \frac{1}{2}(1 + w\delta)\rho - (1 - w\delta)h'$$

gives a bijection $\mathfrak{X}_{\mathrm{Gal}}(\lambda) \to \mathfrak{X}_{\mathrm{Aut}}(\lambda)$*, with inverse given by*

$$h' = \frac{1}{2}(\lambda - \Lambda) \quad \textit{and} \quad \check{\Lambda} = (1 + w\delta)(h + \frac{1}{2}\check{\rho}).$$

3. Examples

Let's see how this plays out for $G = \mathrm{SL}_2$ and $\check{G} = \mathrm{PGL}_2$.

The root data. For $G = \mathrm{SL}_2$, the torus is $H = \mathbb{C}^\times$, with

$$\mathbb{X}^*(H) = \mathbb{Z}\varpi \quad \text{and} \quad \mathbb{X}_*(H) = \mathbb{Z}\check{\alpha}, \quad \text{with} \quad \langle \varpi, \check{\alpha} \rangle = 1.$$

The roots and coroots are $\Phi = \{\pm\alpha\}$ and $\check{\Phi} = \{\pm\check{\alpha}\}$, where $\alpha = 2\varpi$. The coweight lattice is

$$\check{P} = \mathbb{Z}\check{\varpi}, \quad \text{where} \quad \check{\varpi} = \frac{1}{2}\check{\alpha}.$$

The orbits for G**.** According to Proposition 2.3, the G-orbits on X are given by $x = \exp(2\pi i h)\tilde{w}\delta$, where

- For $w = 1$, we have $h \in \mathfrak{h}_{\mathbb{Q}}/\mathbb{X}_*(H) = (\mathbb{Q}/\mathbb{Z})\check{\alpha}$ such that $2h + \check{\rho} \in \check{P}/2\mathbb{X}_*(H)$, i.e., $h = 0, \check{\alpha}/4, \check{\alpha}/2, 3\check{\alpha}/4$.
- For $w = s_\alpha$, we have $h \in \mathfrak{h}_{\mathbb{Q}}/(\mathbb{X}_*(H) + \mathfrak{h}_{\mathbb{Q}})$, i.e., $h = 0$.

The local systems for G**.** According to Proposition 2.3, the characters Λ classifying twisted local systems on the orbits above are:

- For $w = 1$, $\Lambda \in \mathbb{X}^*(H)$ such that $\lambda + \rho - \Lambda = 0$. So these orbits support a unique local system if $\lambda \in \mathbb{X}^*(H) = \mathbb{Z}\varpi$ and none otherwise.
- For $w = s_\alpha$, $\Lambda \in \mathbb{X}^*(H)/2\mathbb{X}^*(H) = (\mathbb{Z}/2\mathbb{Z})\varpi$, with no further condition. So this orbit always supports two local systems independent of λ.

The blocks for G**.** Here is a conceptual summary of the above:

Strong real form	Orbit	x	Λ	Comment
$\mathrm{SU}(2,0)$	Unique	δ	$\lambda+\rho$	Exists if $\lambda \in \mathbb{Z}\varpi$
$\mathrm{SU}(0,2)$	Unique	$\exp(i\pi\check{\alpha})\delta$	$\lambda+\rho$	Exists if $\lambda \in \mathbb{Z}\varpi$
$\mathrm{SL}_2(\mathbb{R})$	Closed	$\exp(i\pi\check{\alpha}/2)\delta$	$\lambda+\rho$	Exists if $\lambda \in \mathbb{Z}\varpi$
	Closed	$\exp(i\pi 3\check{\alpha}/2)\delta$	$\lambda+\rho$	Exists if $\lambda \in \mathbb{Z}\varpi$
	Open	$\tilde{s}_\alpha\delta$	0	Extends if $\lambda \in (2\mathbb{Z}+1)\varpi$
			ϖ	Extends if $\lambda \in 2\mathbb{Z}\varpi$.

When λ is integral, there is one interesting block, which corresponds to the two closed orbits for $\mathrm{SL}_2(\mathbb{R})$ and the local system on the open orbit that extends.

The orbits for $\check{G}$**.** According to Proposition 2.5, the $\check{G}^{alg}$-orbits on $\check{X}_\lambda$ are given by $y = \exp(2\pi i h')\tilde{w}(-\delta)$, where

- For $w = 1$, we have $h' \in \mathfrak{h}^*/(\mathbb{X}^*(H) + \mathfrak{h}^*)$, and $\lambda + \rho \in \mathbb{X}^*(H)$. So this gives a unique orbit if $\lambda \in \mathbb{X}^*(H)$ and none otherwise.
- For $w = s_\alpha$, we have $h' \in \mathfrak{h}^*/\mathbb{X}^*(H)$ such that $\lambda - 2h' \in \mathbb{X}^*(H)/2\mathbb{X}^*(H)$, i.e., $h' = \lambda/2, (\lambda + \varpi)/2$. So this contributes two orbits always.

The local systems for $\check{G}$. According to Proposition 2.5, the characters $\check{\Lambda}$ parametrising local systems on the orbits above are

- For $w = 1$, $\check{\Lambda} \in \check{P}/2\mathbb{X}_*(H) = (\frac{1}{2}\mathbb{Z}/2\mathbb{Z})\check{\alpha}$, i.e., $\check{\Lambda} = 0, \check{\alpha}/2, \check{\alpha}, 3\check{\alpha}/2$. So this orbit supports 4 local systems.
- For $w = s_\alpha$, $\check{\Lambda} = 0$, so these orbits each support a unique local system.

The blocks for $\check{G}$. Here is a conceptual summary of the above:

Real group	Orbit	y	$\check{\Lambda}$	Comment
$\mathrm{PGL}_2(\mathbb{R})$	Open	$-\delta$	0	$\lambda \in \mathbb{Z}\varpi$, always extends
			$\check{\alpha}/2$	$\lambda \in \mathbb{Z}\varpi$, never extends
			$\check{\alpha}$	$\lambda \in \mathbb{Z}\varpi$, always extends
			$3\check{\alpha}/2$	$\lambda \in \mathbb{Z}\varpi$, never extends
	Closed	$\exp(i\pi\lambda)\tilde{s}_\alpha(-\delta)$	0	$\lambda \in (2\mathbb{Z}+1)\varpi$
	Closed	$\exp(i\pi(\lambda+\varpi))\tilde{s}_\alpha(-\delta)$	0	$\lambda \in 2\mathbb{Z}\varpi$
$\mathrm{SO}_3(\mathbb{R})$	Unique	$\exp(i\pi(\lambda+\varpi))\tilde{s}_\alpha(-\delta)$	0	$\lambda \in (2\mathbb{Z}+1)\varpi$
	Unique	$\exp(i\pi\lambda)\tilde{s}_\alpha(-\delta)$	0	$\lambda \in 2\mathbb{Z}\varpi$
$\mathrm{GL}_1(\mathbb{R})$	Unique	$\exp(i\pi(\lambda+\varpi))\tilde{s}_\alpha(-\delta)$	0	$\lambda \notin \mathbb{Z}\varpi$
$\mathrm{GL}_1(\mathbb{R})$	Unique	$\exp(i\pi\lambda)\tilde{s}_\alpha(-\delta)$	0	$\lambda \notin \mathbb{Z}\varpi$

When λ is integral, there is one interesting block, for $\mathrm{PGL}_2(\mathbb{R})$, consisting of the two extendable local systems on the open orbit and the unique local system on the closed orbit.

The correspondence. Here is the bijection between the two sides:

G orbit	x	Λ	$\check{G}$ orbit	y	$\check{\Lambda}$
$\mathrm{SU}(2,0)$	δ	$\lambda+\rho$	$\mathrm{PGL}_2(\mathbb{R})$ open	$-\delta$	$\check{\alpha}/2$
$\mathrm{SU}(0,2)$	$\exp(i\pi\check{\alpha})\delta$	$\lambda+\rho$	$\mathrm{PGL}_2(\mathbb{R})$ open	$-\delta$	$3\check{\alpha}/2$
$\mathrm{SL}_2(\mathbb{R})$ closed	$\exp(i\pi\check{\alpha}/2)\delta$	$\lambda+\rho$	$\mathrm{PGL}_2(\mathbb{R})$ open	$-\delta$	$\check{\alpha}$
$\mathrm{SL}_2(\mathbb{R})$ closed	$\exp(i\pi 3\check{\alpha}/2)\delta$	$\lambda+\rho$	$\mathrm{PGL}_2(\mathbb{R})$ open	$-\delta$	0
$\mathrm{SL}_2(\mathbb{R})$ open	$\tilde{s}_\alpha\delta$	0	$\mathrm{PGL}_2(\mathbb{R})$ closed, λ odd $\mathrm{SO}_3(\mathbb{R})$, λ even $\mathrm{GL}_1(\mathbb{R})$, $\lambda \notin \mathbb{Z}\varpi$	$\exp(i\pi\lambda)\tilde{s}_\alpha(-\delta)$	0
$\mathrm{SL}_2(\mathbb{R})$ open	$\tilde{s}_\alpha\delta$	ϖ	$\mathrm{PGL}_2(\mathbb{R})$ closed, λ even $\mathrm{SO}_3(\mathbb{R})$, λ odd $\mathrm{GL}_1(\mathbb{R})$, $\lambda \notin \mathbb{Z}\varpi$	$\exp(i\pi(\lambda+\varpi))\tilde{s}_\alpha(-\delta)$	0

Note in particular that the interesting blocks on either side match up.

References

[ABV] J. Adams, D. Barbasch, and D. Vogan. *The Langlands classification and irreducible characters for real reductive groups*, Birkhäuser Boston, 1992.