Adams-Barbasch-Vogan explicitly

Aim: Make this ABV bijection completely explicit.

1. Recollection of the statement (geometric version)

2. Combinatorics

3. Example $G = SL_2$, $\check{G} = PGL_2$,

§ 1 : The statement

Fix G, $\check{G}$ dual reductive groups $/\mathbb{C}$

+ pinnings

Convention:

i.e. $H \subset B \subset G$

$\check{H} \subset \check{B} \subset \check{G}$

B and $\check{B}$ have negative roots

+ bases for $\mathfrak{g}_\alpha$, $\check{\mathfrak{g}}_\alpha$ simple root spaces

Dual: $\mathbb{X}^*(H) = \mathbb{X}_*(\check{H})$

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$\mathbb{X}^*(H) \supset \Phi = \{\text{roots for } G\}$

$= \{\text{coroots for } \check{G}\}$

$\mathbb{X}_*(H) \supset \check{\Phi} = \{\text{coroots for } G\}$

$= \{\text{roots for } \check{G}\}$

Inner class:

$\delta : G \to G$ involution fixing the pinning

($\Leftrightarrow$ involution of the root datum)

$\rightsquigarrow$ $\check{\delta} : \check{G} \to \check{G}$ —— " ——

such that $\check{\delta} = -w_0 \delta \; (= -w_0 \circ \delta)$ on root data

w_0 = longest elt of Weyl group W

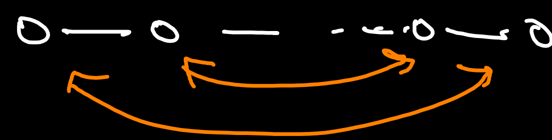
Extended groups:

$G_\Gamma = G \rtimes \{1, \delta\}$, $\check{G}_\Gamma = \check{G} \rtimes \{1, \check{\delta}\}$.

Example:

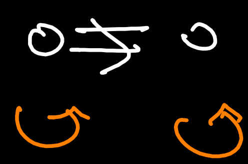
$\delta = \mathrm{id}$

$G = SL_n$



$-w_0 = -w_0\delta$

$G = Sp_4$



$-w_0$

Automorphic (G) side

Define

$$X = \{ x \in G_\Gamma - G \mid x^2 \in Z(G) \text{ has finite order} \}$$

$X \times \mathcal{B}$

G/B

$\tilde{\mathcal{B}} \longrightarrow \mathcal{B}$ H-torsor

$\tilde{\mathcal{B}} = G/N$

$\mathcal{B} = G/B$

$\Rightarrow$ H-torsor

$$\tilde{X} := \tilde{\mathcal{B}} \times_{\mathcal{B}} X \xrightarrow{\pi} X.$$

For $\lambda \in \mathfrak{h}^*$,

$\Rightarrow$

$$\mathcal{D}_{X,\lambda} = \pi_*(\mathcal{D}_{\tilde{X}})^H \otimes_{S(\mathfrak{h})} \mathbb{C}_{\lambda-\rho}.$$

$$\mathfrak{X}_{Aut}(\lambda) := \left\{ \begin{array}{l} \text{irreducible objects} \\ \text{in } HC(\mathcal{D}_{X,\lambda}, G) \end{array} \right\}$$

$$\cong \left\{ (Q, \gamma) \;\middle|\; \begin{array}{l} Q \subset X \text{ } G\text{-orbit} \\ \gamma \text{ irred.} \\ \text{equivariant} \\ \lambda\text{-twisted loc sys} \end{array} \right\}$$

<u>Relationship to real grps</u>

$$G \backslash X = \coprod_x G^{\theta_x} \backslash \mathcal{B}, \quad \theta_x = Ad_x$$

So

$$HC(\mathcal{D}_{X,\lambda}, G) = \bigoplus_x HC(\mathcal{D}_{\mathcal{B},\lambda}, G^{\theta_x})$$

By Beilinson-Bernstein,

if λ is integrally dominant then

$$\Gamma : HC(\mathcal{D}_{\mathcal{B},\lambda}, G^{\theta x}) \to HC(\mathfrak{g}, G^{\theta x})_\lambda$$

is the quotient by a Serre subcategory.

If λ is regular, then it's an equivalence.

$$\left\{\begin{matrix}\text{strong}\\ \text{invs}\end{matrix}\right\} = \coprod_x G\cdot x = \coprod_x G/\mathrm{Stab}_G(x)$$

$$= \coprod_x G/G^{\theta x}$$

$$G\backslash X = \coprod_x \underbrace{G\backslash\left(G/G^{\theta x} \times \mathcal{B}\right)}_{= G^{\theta x}\backslash\mathcal{B}}$$

Galois side

Keep $\lambda \in \mathfrak{h}^* = \check{\mathfrak{h}}$ fixed.

Define

$$\check{X}_\lambda = \{ y \in \check{G}_r - \check{G} \mid y^2 = \exp(2\pi i \lambda) \} \times \check{\mathcal{B}}_\lambda$$

where $\check{\mathcal{B}}_\lambda = \check{G}_\lambda / \check{B}_\lambda$

$$\check{G}_\lambda = Z_{\check{G}}(\exp(2\pi i \lambda))$$

$$\check{B}_\lambda = \check{G}_\lambda \cap \check{B}.$$

Let

$\check{G}^{alg}$ = pro-algebraic universal cover of $\check{G}$

$$\check{G}^{alg}_\lambda = \check{G}^{alg} \times_{\check{G}} \check{G}_\lambda .$$

$$\mathcal{X}_{Gal}(\lambda) := \left\{ \begin{array}{l} \text{irreducibles in} \\ HC(\mathcal{D}_{\check{X}_\lambda}, \check{G}_\lambda^{alg}) \end{array} \right\}$$

$$\cong \left\{ (\check{Q}, \check{\gamma}) \;\middle|\; \begin{array}{l} \check{Q} \subset \check{X}_\lambda \text{ orbit} \\ \check{\gamma} \text{ equiv.} \\ \text{local system} \end{array} \right\}$$

<u>Note:</u>

$$\check{G}_\lambda^{alg} \backslash \check{X}_\lambda = \bigsqcup_y (\check{G}_\lambda^{\theta_y})^{alg} \backslash \check{B}_\lambda$$

where $(\check{G}_\lambda^{\theta_y})^{alg} = \check{G}_\lambda^{\theta_y} \times_{\check{G}} \check{G}^{alg}$

$\theta_y = Ad_y$.

<u>Rel^n w/ ABV</u>: Assume λ integrally dominant.

$\rightsquigarrow \check{B}_\lambda \subset \check{P}_\lambda \subset \check{G}_\lambda$

roots $\{ \check{\alpha} \in \check{\Phi} \mid \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}_{\leq 0} \}$

$\rightsquigarrow \check{X}_\lambda^{\mathrm{par}} = \{ y \in \check{G}_\Gamma - \check{G} \mid y^2 = \exp(2\pi i \lambda) \}$

$\times \; \check{G}_\lambda / \check{P}_\lambda$.

$\rightsquigarrow$ Full subcategory

$$HC(\mathcal{D}_{\check{X}_\lambda^{\mathrm{par}}}, \check{G}_\lambda^{\mathrm{alg}}) \hookrightarrow HC(\mathcal{D}_{\check{X}_\lambda}, \check{G}_\lambda^{\mathrm{alg}})$$

__Thm__ [ABV, Thms 1.18, 1.24]

(1) There is a bijection

$$LLC: \mathfrak{X}_{Aut}(\lambda) \xrightarrow{\sim} \mathfrak{X}_{Gal}(\lambda)$$

(2) The following two pairings

$$K(HC(\mathcal{D}_{X,\lambda}, G)) \otimes K(HC(\mathcal{D}_{\check{X}_\lambda}, \check{G}_\lambda^{\mathrm{alg}})) \longrightarrow \mathbb{Z}$$

coincide:

$$\langle [j_{!*}\gamma], [j_{!*}\check{\gamma}] \rangle = (-1)^{\ell(\check{\gamma})} e(\check{\gamma}) \cdot \delta_{\gamma, LLC(\check{\gamma})}$$

and

$$\langle [j_!\gamma], [j_!\check{\gamma}] \rangle = (-1)^{\ell(\check{\gamma})} e(\check{\gamma}) \cdot \delta_{\gamma, LLC(\check{\gamma})}.$$

where

$\ell(\check{\gamma}) = \ell(\check{\gamma}, \check{Q}) = \dim \check{Q}$

$e(\check{\gamma})$ = Kottwitz's sign

(3) If λ is integrally dominant the kernel of

$$K(HC(\mathcal{D}_{X,\lambda}, G)) \longrightarrow \bigoplus_x K(HC(\mathfrak{g}, G^{\theta_x})_\lambda)$$

is orthogonal complement to

$$K(Hc(\mathcal{D}_{\check{X}_\lambda^{per}}, \check{G}_\lambda^{alg}))$$

$$\subset K(Hc(\mathcal{D}_{\check{X}_\lambda}, \check{G}_\lambda^{alg})).$$