Ref:

[1] [Oblomkov-Yun]: Geometric reps of graded & rat'l Cherednik algebras

[2] Rep. theory Seminar 2017 notes.

Goal: Relation of affine Spr fibers & Hitchin fibers

X alg curve/k. $\mathfrak{g}/\!/G$

$a: X \to \overset{\shortparallel}{\mathfrak{c}}/\mathbb{G}_m$

Follow § 6.6 of [1].)

$U_a = \text{Preimage}[\mathfrak{c}^{rs}/\mathbb{G}_m] \subseteq X$

$X - U_a$ finite many pts.

Analogous: The product formula in [Ngô: fund. lemma Prop 4.15.1].

$$\mathcal{P}_a^{gl.} \overset{\prod_{x \in X - U_a} \mathcal{P}^{loc}_{a_x}}{\times} \left(\prod_{x \in X - U_a} \mathrm{Aff\,Spr}_{a_x} \right) \longrightarrow \mathrm{Hitch}_a$$

homeom of stacks

I) $Sp_{\mathbb{P}, \nu}$

G almost simple connected simply conn. red. gp/$\mathbb{C}$

$F = \mathbb{C}((t))$
$\cup\!\shortmid$
$\mathcal{O} = \mathbb{C}[[t]]$
$\shortparallel$
$\mathcal{O}_F$

Kari's lecture

Rmk: In [1], $F_n = \mathbb{C}((t^{1/n}))$ field ext. of F.

$$F_\infty = \bigcup_{n \geq 1} F_n$$

For: $e = 1$ or 2 or 3

$\mathbb{G}$ = type A, D $\quad$ E_6 $\quad$ D_4.

Fix $\theta : \mu_e \hookrightarrow \mathrm{Out}(\mathbb{G})$

Then: $G(F)$: group scheme over F

$$\simeq \left(\mathrm{Res}_F^{F_e} (\mathbb{G} \otimes_{\mathbb{C}} F_e) \right)^{\mu_e},$$

where $\mu_e \curvearrowright \mathbb{G}$ by out. morp.

$\mu_e \curvearrowright F_e$ by Galois action

In my talk; will take $e = 1$

and: $G(F) = \mathbb{G} \otimes \mathbb{C}((t))$.

Let $\mathbb{P} \subseteq G(F)$ standard parahoric subgp.

that is: • connected gp subscheme of $G(F)$

• finite codimension

• I = Iwahori subgp $\subseteq \mathbb{P}$

Recall: Fix $B \subseteq G$ Borel.

$$\pi : G(\mathcal{O}) \longrightarrow G$$
$$t \longmapsto 0$$
$$I \subseteq G(\mathcal{O}) := \pi^{-1}(B).$$

Ex: $SL_2(F)$.

Parahoric. $G(\mathcal{O}) = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}$, or $\begin{bmatrix} \mathcal{O} & t^{-1}\mathcal{O} \\ t\mathcal{O} & \mathcal{O} \end{bmatrix}$.

Up to $G(F)$-conj: $I = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \mathcal{O} \end{bmatrix}$

Affine Springer fiber at $\gamma \in \mathfrak{g}(F) = \mathrm{Lie}\, G(F)$

$$Sp_{\mathbb{P},\gamma} := \left\{ g\mathbb{P} \in G(F)/\mathbb{P} \;\middle|\; \mathrm{Ad}(g^{-1})\gamma \in \mathrm{Lie}(\mathbb{P}) \right\}$$

$$\chi : \mathfrak{g} \longrightarrow \mathfrak{g}/\!/G = \mathfrak{t}/\!/W =: \mathfrak{C}$$

$$\rightsquigarrow \quad \chi : \mathfrak{g}(F) \longrightarrow \mathfrak{C}(F)$$
$$\gamma \longmapsto a$$

<u>Kostant section:</u>

$$\kappa : \mathfrak{C}(F) \hookrightarrow \mathfrak{g}(F)$$
$$a \longmapsto \gamma$$

<u>Two torus actions:</u> $\quad \mathbb{G}_m^{rot}, \quad \mathbb{G}_m^{dil}$

- $\mathbb{G}_m^{rot} \curvearrowright F = \mathbb{C}((t))$ by scalling t.

- $\mathbb{G}_m^{dil} \curvearrowright \mathfrak{g}(F)$ by $(\lambda, X) \mapsto \lambda X$

$\Rightarrow \mathbb{G}_m^{rot} \times \mathbb{G}_m^{dil} \curvearrowright \mathfrak{g}(F), \mathcal{C}(F)$.

Fix $\nu \in \mathbb{Q}$ rat' number (slope). $\boxed{\text{In } RCA_{\hbar, c}, \text{ take } \hbar \text{ to be } 1 \text{ \& } c = \nu.}$

$\nu = d/m$ $\quad \mathbb{G}_m(\nu) \subseteq \mathbb{G}_m^{rot} \times \mathbb{G}_m^{dil}$ s.t.: character gp:

$s \longmapsto (s^m, s^{-d})$, $\quad 0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to X^*(\mathbb{G}_m(\nu)) \to 0$

$\mathbb{G}_m(\nu) \to (G^{ad}(F) \rtimes \mathbb{G}_m^{rot}) \times \mathbb{G}_m^{dil}$ $\quad 1 \longmapsto (d, m) = \nu$

$s \longmapsto (s^{d\rho^\vee}, s^m, s^{-d})$

<u>Define</u>: $\mathcal{C}^{rs}(F)_\nu = \{ a \in \mathcal{C}(F) \mid$ • a regular semisimple, • a fixed under $\mathbb{G}_m(\nu)$ $\}$

homogeneous of slope ν.

(ie: $s^d \cdot a(t) = a(s^m t)$ for all $s \in \mathbb{C}^*$)

For $a \in \mathcal{C}^{r.s}(F)$, view $a : \operatorname{Spec} F \to \mathcal{C}^{rs}_F$.

$\rightsquigarrow$ $\pi_a : \mathrm{Gal}(F_\infty/F) \cong \varprojlim_n \mu_n = \hat{\mathbb{Z}}(1) \longrightarrow W$

Thm 3.2.5: $\mathcal{C}^{r.s}(F)_v \overset{1:1}{\longleftrightarrow} \mathrm{Reg}(\hat{\mathbb{Z}}(1) \to W)/W$

$\pi : \hat{\mathbb{Z}}(1) \to W$ Let $X \in \mathfrak{t}_v^{r.s}$. $\mathbb{Q}/\mathbb{Z} \ni v$

$\mathfrak{t} = \bigoplus \mathfrak{t}_\xi$

$\xi \in \mathbb{Q}/\mathbb{Z}$

eigenspace decomposi under $\hat{\mathbb{Z}}(1)$

Define $a = \chi(Xt^v) \in \mathcal{C}^{r.s}(F_\infty)$

Regularity of $\pi \Rightarrow a \in \mathcal{C}^{r.s}(F)$.

Def: a is called elliptic, if $\mathfrak{t}^{\pi_a(\hat{\mathbb{Z}}(1))} = 0$.

<u>Will Focus on:</u> $\quad \gamma = \kappa(a)$

$$\mathrm{Sp}_{\mathbb{P},a} := \mathrm{Sp}_{\mathbb{P},\gamma}, \quad a \in \mathfrak{C}^{\gamma.s}(F)_v.$$

Symmetry: $\quad \mathbb{G}_m(v)$ fixes $\gamma \Rightarrow \mathbb{G}_m(v) \curvearrowright \mathrm{Sp}_{\mathbb{P},a}$.

$$G_\gamma := \{ g \in G(F) \mid \mathrm{ad}(g^{-1})\gamma = \gamma \} \curvearrowright \mathrm{Sp}_{\mathbb{P},a}$$

$$G_\gamma \rtimes \mathbb{G}_m(v) \curvearrowright \mathrm{Sp}_{\mathbb{P},a}.$$

In §3.3.7: Local Picard gp:

Assume: $\nu > 0$, admissible slope s.t:

$$: \mathfrak{C}(F)^{r,s}_{\nu} \subseteq \mathfrak{C}(\mathcal{O}_F)$$

$$\begin{matrix} \ni & & \ni & \mathcal{J} \\ a & & a & \downarrow \end{matrix}$$

Think: $a : \operatorname{Spec} \mathcal{O}_F \longrightarrow \mathfrak{C}$

Let I over $\mathfrak{g}$: universal centr. gp scheme $\rightsquigarrow I|_{\mathfrak{C}} =: J$

Regular centralizer gp scheme J over $\mathfrak{C}$

$$J_a := a^* J \quad \text{gp scheme over } \mathcal{O}_F.$$

- Commutative gp scheme P^{loc}_{ν} over $\mathfrak{C}(F)^{r,s}_{\nu}$:

$$\ni \\ a$$

fiber $P^{loc}_a := J_a(F) / J_a(\mathcal{O})$ local Picard gp.

Note: lemma 5.2.5: $Gr(F) \curvearrowright Sp_{P,r}$ factors through P_a.

That is: $J_a(F) \cong Gr(F)$ & $J_a(\mathcal{O})$ acts trivially on $Sp_{P,r}$

Recall:

$F = \mathbb{C}((t))$ $\qquad$ G reductive gp $/\mathbb{C}$.

$F_n = \mathbb{C}((t^{\frac{1}{n}}))$ $\qquad$ $G(F) = G \otimes_{\mathbb{C}} \mathbb{C}((t))$.

$F_\infty = \bigcup_n F_n$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\chi"] & \mathfrak{g}/\!/G = \mathfrak{t}/\!/W = \mathfrak{C} \arrow[l, bend left, "\kappa"] \end{tikzcd}$$

$$\begin{array}{ccc} \mathfrak{g}(F) & \longrightarrow & \mathfrak{C}(F) \\ \cup & & \cup \\ \gamma & \longleftarrow\!\shortmid & a \end{array}$$

Fix $\nu = \frac{d}{m} \in \mathbb{Q}_{>0}$ slope $\qquad$ Fix $\mathbb{P} \subseteq G(F)$ parahoric e.g. $\mathbb{P} = G(\mathcal{O})$ or I,

$Sp_\nu \rightsquigarrow$ Fiber $Sp_a := Sp_\gamma = \{ g \in G(F)/\mathbb{P} \mid Ad(g^{-1})\gamma \in Lie(\mathbb{P}) \}$

$\downarrow$

$\mathfrak{C}_\nu^{r.s}(F) = \{ a \in \mathfrak{C}(F) \mid \cdot\ a \text{ regular. ss} \quad \cdot\ a \text{ fixed by } \mathbb{G}_m(\nu) \}$

Clarification: 1) $\mathbb{G}_m(v) \longrightarrow (G^{ad}(F) \rtimes \mathbb{G}_m^{rot}) \times \mathbb{G}_m^{dil}$ 8.2

$$s \longmapsto (s^{d\rho^\vee}, \quad s^{m}, \quad s^{-d})$$

e.g.: $\gamma = \begin{bmatrix} 0 & 1 \\ t^d & 0 \end{bmatrix} \in \mathrm{Lie}(PGL_2(F))$.

$$s^{d\rho^\vee} = \begin{bmatrix} 1 & 0 \\ 0 & s^d \end{bmatrix}$$

$$s^{-d\rho^\vee} \gamma\, s^{d\rho^\vee} = \begin{bmatrix} 1 & 0 \\ 0 & s^{-d} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ t^d & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s^d \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ s^{-d}t^d & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s^d \end{bmatrix}$$

$$= \begin{bmatrix} 0 & s^d \\ s^{-d}t^d & 0 \end{bmatrix}$$

Scale t by $s^2 t$

$$\begin{bmatrix} 0 & s^d \\ s^d t^d & \end{bmatrix} \triangle = s^d \begin{bmatrix} 0 & 1 \\ t^d & 0 \end{bmatrix}$$

8.3

② **Def**: F-torus T: an alg gp$/F$, s.t

$T(\bar{F}) \cong (\bar{F}^{\times})^n$. , $\bar{F}$ alg closure.

T is split$/_F$ if $T(F) \cong (F^{\times})^n$.

[GKM: Goresky - Kottwitz - Macpherson] §5.2 5.3.

$F = \mathbb{C}((t)) \leq F_s \leq \bar{F}$ $\qquad A \leq G$

separable closure $\qquad$ max torus

weyl gp

$$\{\text{Max. } F\text{-tori in } G\}/_{G(F)\text{-conj}} \xleftrightarrow{1:1} H^1(F, W) = H^1_{ét}(\operatorname{Spec} F, W) \cong H^1(\operatorname{Gal}(F_s/F), W)$$

Recall: $H^1_{et}(\operatorname{Spec} F, M(F)) = H^1(\operatorname{Gal}(F_s/F), M(F_s))$ has an action of $\operatorname{Gal}(F_s/F)$

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Assume: group$/F$ $= \{ f : \operatorname{Gal}(F_s/F) \to M \mid f(gh) = (g \cdot f(h)) \bullet f(g) \} \Big/$ 1-cocycles

$g \mapsto \underset{\hat{M}}{(g\cdot a)} \cdot \underset{\hat{M}}{a^{-1}}$ for some $a \in M$.

If $\operatorname{Gal}(F_s/F) \curvearrowright M(F_s)$ trivial

$$\Rightarrow H^1(F, M) = \operatorname{Hom}(\operatorname{Gal}(F_s/F), M(F_s)).$$

Now

T : F-tors in $G(F)$

$\Rightarrow \exists\, h \in G(F_s)$

s.t. $T = h A h^{-1}$.

$= \sigma(h) A \sigma(h)^{-1}$

$\sigma \in \operatorname{Gal}(F_s/F)$

Define $\phi : \operatorname{Gal}(F_s/F) \to G(F_s)$

$\sigma \longmapsto h^{-1} \sigma(h)$.

1-cocycle

$\lfloor 8.5$

Eg: $\gamma = \begin{bmatrix} 0 & 1 \\ t^d & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -t^{-\frac{d}{2}} & t^{-\frac{d}{2}} \\ 1 & 1 \end{bmatrix}}_{= h} \begin{bmatrix} -t^{d/2} & 0 \\ 0 & t^{d/2} \end{bmatrix} \underbrace{\begin{bmatrix} -\frac{1}{2}t^{\frac{d}{2}} & \frac{1}{2} \\ \frac{1}{2}t^{\frac{d}{2}} & \frac{1}{2} \end{bmatrix}}_{= h^{-1}}$.

$Gal(F_2/F) = \mathbb{Z}_2 \ni \sigma$ s.t. $\sigma(t^{\frac{1}{2}}) = -t^{1/2}$.

$\Rightarrow \sigma \begin{bmatrix} -t^{-\frac{d}{2}} & t^{-\frac{d}{2}} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} t^{-\frac{d}{2}} & -t^{-\frac{d}{2}} \\ 1 & 1 \end{bmatrix}$

$h^{-1}\sigma(h) = \begin{bmatrix} -\frac{1}{2}t^{\frac{d}{2}} & \frac{1}{2} \\ \frac{1}{2}t^{\frac{d}{2}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} t^{-\frac{d}{2}} & -t^{-\frac{d}{2}} \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in W$

$\begin{bmatrix} 0 & 1 \\ t^d & 0 \end{bmatrix} = h \begin{bmatrix} -t^{d/2} & \\ & t^{d/2} \end{bmatrix} h^{-1}$

$= \sigma(h) \begin{bmatrix} t^{d/2} & \\ & -t^{d/2} \end{bmatrix} \sigma(h)^{-1}$

$\Rightarrow (h^{-1}\sigma(h))^{-1} \begin{bmatrix} -t^{d/2} & \\ & t^{d/2} \end{bmatrix} h^{-1}\sigma(h)$

$= \begin{bmatrix} t^{d/2} & \\ & -t^{d/2} \end{bmatrix}$

$\Rightarrow h^{-1}\sigma(h) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in W.$

clearly: $h^{-1}\sigma(h) \in N_G(A)$ & $\sigma \longmapsto h^{-1}\sigma(h)$ coboundary $h \in G(F_s)$.

8.6

$$\left(\begin{array}{l} \text{since } T = hAh^{-1} \\ \qquad = \sigma(h) A \sigma(h)^{-1} \\ \qquad \Rightarrow (h^{-1}\sigma(h))^{-1} A\, h^{-1}\sigma(h) \\ \qquad = A. \end{array}\right)$$

$$\Rightarrow T \rightsquigarrow \ker\left[H^1(F, N_G(A)) \to H^1(F, G(F))\right]$$

$$1 \to A \to N_G(A) \to W \to 1$$

$$\to H^1(F, A) \to H^1(F, N_G(A)) \xhookrightarrow{\text{inj.}} H^1(F, W)$$

$H^1(F, A) = 0$ since for our field F, $H^1(F, T') = 0$ for any F-torus T'!

$$\Rightarrow \{\text{max tori in } G(F)\}/\sim \;\hookrightarrow\; H^1(F\, W).$$

Rmk: Split tori $\longmapsto$ trivial map.

8.7

On the other hand.

$\forall$ 1 cocycle $\quad \mathrm{Gal}(F_s/F) \longrightarrow W$

$\rightsquigarrow \mathrm{Gal}(F_{|w|}/F) \longrightarrow W$

cyclic order $|w|$

generator $\tau_{|w|} : t^{\frac{1}{|w|}} \to \zeta_{|w|}\, t^{\frac{1}{|w|}}$

$$\tau_{|w|} \longmapsto w$$

lift $w \in W$ to $\dot{w} \in N_G(A)$ $\quad$ Let $l := \mathrm{ord}(\dot{w})$

$$\Rightarrow \quad \mathrm{Gal}(F_l/F) \longrightarrow N_G(A)$$

$$\tau_l \longmapsto \dot{w}$$

$\dot{w}$ semisimple $\Rightarrow$ $\exists$ max torus A' in $G(\mathbb{C})$
$\in G(\mathbb{C})$ $\quad$ s.t. $\dot{w} \in A'(\mathbb{C})$.

By Hilbert's thm 90 $\Rightarrow H^1(\mathrm{Gal}(F_\ell/F),\ A'(F_\ell)) = \{1\}$. 88

$A'(F_\ell)$ split torus $/F_\ell$

$\Rightarrow \exists$ an elt $b \in A'(F_\ell)$ s.t

$$b^{-1}\tau_1(b) = \dot{w}.$$

Define the max. torus

$$T := b A b^{-1}.$$ gives the explicit construction of T.

In fact: b is explicit:

$\dot{w} \in A'(k)$ has order ℓ.

$\Rightarrow \exists\ \mu \in X_*(A') = \mathrm{Hom}(\mathbb{G}_m, A')$, s.t $\mu(\zeta_\ell^{-1}) = \dot{w}$

Set $b := \mu(t^{\frac{1}{\ell}})^{-1}$ then: $b^{-1}\tau_1(b) = \dot{w}$

In [OY] Thm 3.25:

$$\mathcal{C}(F)^{r.s}_{v} \overset{\text{bij.}}{\longleftrightarrow} \mathrm{Reg}(W)_{\bar{v}}/W.$$

$$\mathrm{Reg}(W)_{\bar{v}} = \left\{ (\pi, X) \;\middle|\; \pi : \hat{\mathbb{Z}}(1) \to W \text{ regular hom.} \quad X \in \mathfrak{t}_{\bar{v}} \cap \mathfrak{t}^{r.s} \right\}$$

For $\gamma \in \mathfrak{g}(F)$

Let $Y := \gamma(1) \in \mathfrak{g}^{r.s}_{\bar{v}}$ (eigenvector under $\hat{\mathbb{Z}}(1)$).

$\mathfrak{g}_Y :=$ the centralizer of Y in $\mathfrak{g}$ (normalized by $\hat{\mathbb{Z}}(1) \curvearrowright \mathfrak{g}_Y$) $/W$-sim. conj.

Choose $g \in G^{ad}$ s.t. $\mathrm{ad}_g(\mathfrak{g}_Y) = \mathfrak{t}$

$\Rightarrow$ We get a homomorphism ?

$$\pi_g : \mu_m \longrightarrow N_{G^{ad}}(\mathfrak{g}_Y) \cong N_{G^{ad}}(\mathfrak{t}) \twoheadrightarrow W$$

$$\delta \longmapsto g^{-1}\delta(g)$$

& the vector.

$$X_g := \mathrm{ad}(g)\, Y \in \mathfrak{t}^{r.s}_{\bar{v}}.$$

$$\Rightarrow \quad (\pi_g, X_g) \in \mathrm{Reg}(W)_{\bar{v}}/W.$$

8.10

Symmetry on Sp_γ : $\circlearrowleft$ $G_\gamma(F) \rtimes \mathbb{G}_m(\gamma)$

G_γ — torus

$$Sp_\gamma \downarrow \mathfrak{C}^{r.s}_\gamma(F).$$

<u>Assume</u>: γ addmissible

$\Rightarrow$ $a \in \mathfrak{C}^{r.s}_\gamma(F) \subseteq \mathfrak{C}^{r.s}_\gamma(\mathcal{O})$

$$\begin{array}{c} J \\ \downarrow \\ a : \operatorname{Spec}\mathcal{O}_F \longrightarrow \mathfrak{C} \end{array}$$

J — universal centralizer gp scheme.

$$\left\{\begin{array}{l} J_a(\mathcal{O}) = a^*J \quad \text{on } \operatorname{Spec}(\mathcal{O}) \\ J_a(F) \cong G_\gamma(F). \end{array}\right.$$

<u>Lemma 5.2.5</u>: The action of $G_\gamma(F)$ on Sp_γ factors through

$$P_a^{loc} := J_a(F)/J_a(\mathcal{O}) \rightsquigarrow \mathcal{P}_\gamma$$

comm. gp scheme / $\mathfrak{C}^{rs}_\gamma$.

II) Hitchin fibers:

$X =$ weighted proj. line $\mathbb{P}(m, 1)$

$= [\mathbb{A}^2 \setminus \{(0,0)\} / \mathbb{G}_m]$ quotient stack.

$\mathbb{G}_m$ action: by weight $(m, 1)$.

$0 = [0, 1]$ no non-trivial automorph.

$\infty = [1, 0]$ has auto. μ_m.

Identify: $\deg : \mathrm{Pic}(X) \longrightarrow \frac{1}{m}\mathbb{Z}$

$\mathcal{O}_X(\nu) \longleftarrow\!\shortmid \nu$

For $\mathcal{L}$, $\deg \mathcal{L} > 0$.

$$H^0(X, \mathcal{L}) = \mathbb{C}[\xi, \eta]_{m \deg \mathcal{L}}$$

hom. polys in ξ, η, total weight $m \cdot \deg \mathcal{L}$.

$\mathbb{G}_m^{rot} \curvearrowright X \qquad (t, [\xi, \eta]) \longmapsto (t\xi, \eta)$.

Fix $\mathbb{P} \subseteq G(F)$

$\rightsquigarrow$ group scheme $\begin{array}{c} \mathcal{G}_{\mathbb{P}} \\ \downarrow \\ X \end{array}$ two opens: $V = X \setminus \{0\}$

$\hat{\mathcal{O}}_0 = \mathbb{C}[[t]]$.

gluing $\quad G \times V \quad$ over $V = X \setminus \{0\}$ $\quad$ and $\quad \mathbb{P} \quad$ over $\hat{\mathcal{O}}_0$.

along: $\quad \operatorname{Spec} \hat{K}_0 = \operatorname{Spec} \mathbb{C}((t))$.

How to glue?

Both $G \times V \big|_{\operatorname{Spec} F}$ & $\mathbb{P} \big|_{\operatorname{Spec} F}$ are canonically isomorphic to the group scheme $G \times \operatorname{Spec} F$.

Fix $\mathbb{P} \subseteq G(F)$ $\quad \mathcal{L} \to X$ line bundle.
($\cong \mathcal{O}(\nu)$ later).

Def: $\mathcal{M}_{\mathbb{P}}^{Hit} := \{ (\mathcal{E}, \varphi) \mid$ • $\mathcal{E}$: $\mathcal{G}_P$-torsor over X.
• $\varphi \in H^0(X, \mathrm{Ad}_{\mathcal{G}_P}(\mathcal{E}) \otimes \mathcal{L}) \}$

$$\mathrm{Ad}_{\mathcal{G}_P}(\mathcal{E}) \otimes \mathcal{L} = \left(\mathcal{E} \times_{\mathcal{G}_P} \mathrm{Lie}(\mathcal{G}_P)\right) \otimes \mathcal{L}$$

Hitchin fibration:

$$\mathcal{M}_{\mathbb{P}}^{Hit} \longrightarrow \mathcal{A} = \bigoplus_{i=1}^{r} H^0(X, \mathcal{L}^{\otimes d_i}) = \bigoplus_{i=1}^{r} \mathbb{C}[\xi, \eta]_{d_i \cdot m \deg(\mathcal{L})}$$

will explain

Intrinsic :

$$\chi : \mathfrak{g} \to \mathfrak{C} = \operatorname{Spec} \operatorname{Sym}(\mathfrak{g}^*)^G \quad \circlearrowleft \quad G, \mathbb{G}_m^{\mathrm{dil}} \text{-equiv.}$$

$G \curvearrowright \mathfrak{C}$ trivial.

$$\rightsquigarrow \quad \begin{array}{c} \mathrm{Ad}(\mathcal{E}) \\ \| \\ \mathcal{E} \times_G \mathfrak{g} \end{array} \longrightarrow \mathcal{E} \times_G \mathfrak{C} = X \times \mathfrak{C}.$$

twist by $\mathcal{L}$

$$\rightsquigarrow \quad \mathrm{Ad}(\mathcal{E}) \otimes \mathcal{L} \longrightarrow X \times (\mathfrak{C} \otimes \mathcal{L}) =: \mathfrak{C}_{X,\mathcal{L}}$$

is not a vector bundle

But just a bundle.

$$\rightsquigarrow \quad \begin{array}{c} H^0(X, \mathrm{Ad}(\mathcal{E}) \otimes \mathcal{L}) \\ \cup\!\!\!| \\ \varphi \end{array} \xrightarrow{\chi_{\mathcal{E},\mathcal{L}}} H^0(X, \mathfrak{C}_{X,\mathcal{L}}) = \mathcal{A}.$$

Hitchin map $\quad (\mathcal{E}, \varphi) \longmapsto \chi_{\mathcal{E},\mathcal{L}}(\varphi)$

Non-canonical:

Recall: $\mathscr{C} = \mathrm{Spec}\ \mathrm{Sym}(\mathfrak{g}^*)^G = \mathrm{Spec}(\mathfrak{t}^*)^W$

$\mathrm{Sym}(\mathfrak{g}^*)^G = \mathrm{Sym}(\mathfrak{t}^*)^W = $ Poly ring in r variables

$\circlearrowleft$

$f_1, \dots, f_r$ homog. generator

deg: $d_1, \dots, d_r$.

View: $f_i : \mathfrak{g}^{\otimes d_i} \longrightarrow \mathbb{C}$ G-invariant polynomial

For any G-torsor $\mathcal{E}$ over X.

$f_i : \mathrm{Ad}(\mathcal{E})^{\otimes d_i} \longrightarrow \mathcal{O}_X$.

$\rightsquigarrow f_i^{\mathcal{L}} : (\mathrm{Ad}(\mathcal{E}) \otimes \mathcal{L})^{\otimes d_i} \longrightarrow \mathcal{L}^{\otimes d_i}$.

Let φ Higgs field on $\mathcal{E}$: $\varphi \in H^0(\mathrm{Ad}(\mathcal{E}) \otimes \mathcal{L})$.

Evaluate $f_i^{\mathcal{L}}$ on the section $\varphi^{\otimes d_i}$ of $(\mathrm{Ad}(\mathcal{E}) \otimes \mathcal{L})^{\otimes d_i}$.

get: $f_i^{\mathcal{L}}(\varphi^{\otimes d_i}) \in H^0(X, \mathcal{L}^{\otimes d_i})$.

<u>E.g.</u>: $G = GL_n$. $\mathrm{Sym}(\mathfrak{t}^*)^W = \mathbb{C}[e_1, \dots, e_n]$

$f_i = e_i =$ elementary sym. poly

deg $1, 2, \dots, n$.

$f_i^{\mathcal{L}}(\varphi^{\otimes d_i}) =$ coeff in char. poly of φ.

coeff in $\det(tI - \varphi) = 0$.

Consider the locus:

$$\mathcal{A}^{\heartsuit} \subseteq \mathcal{A} = \{ a : X \to [\mathfrak{C}/\mathbb{G}_m] \}$$

$$\text{\rotatebox{90}{$\cong$}}$$

$$\{ a : \quad \text{generic pt} \mapsto [\mathfrak{C}^{rs}/\mathbb{G}_m] \}$$

<u>Two torus actions:</u>

$$\mathbb{G}_m^{rot} \curvearrowright X \text{ by: } (t, [\xi, \eta]) \longmapsto (t\xi, \eta)$$

$$\xrightarrow{\text{induces}} \mathbb{G}_m^{rot} \curvearrowright \mathcal{A} \qquad \mathcal{M}_{\mathbb{P}}^{Hit}$$

$$\downarrow$$

$$\mathcal{A} \ni f(\xi, \eta) \longmapsto f(t\xi, \eta).$$

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$\mathbb{G}_m^{dil} \curvearrowright \mathcal{M}_{\mathbb{P}}^{Hit} \ni (\mathcal{E}, \varphi) \mapsto (\mathcal{E}, t\varphi)$

$$\begin{array}{c} \mathcal{M}_{\mathbb{P}}^{Hit} \\ \downarrow \\ \mathcal{A} \end{array} \curvearrowleft \text{ by weights } d_1, d_2, \dots d_r.$$

<u>Def.</u>: $a \in \mathcal{A}$ is homoge. of slope ν if it is fixed by

$$\mathbb{G}_m(\nu) \subseteq \mathbb{G}_m^{rot} \times \mathbb{G}_m^{dil}.$$

Focus on: $f_{\mathbb{P},\nu}: \mathcal{M}_{\mathbb{P},\nu}^{\heartsuit} \longrightarrow \mathcal{A}_\nu^{\heartsuit} \xrightarrow{\cong} \mathfrak{c}_\nu^{rs}$

$$a \longmapsto a|_{\operatorname{Spec} \hat{F}_0}.$$

Symmetry: on Hitchin: §6.3.6 : $\mathcal{P}_a \curvearrowleft$ global $\mathbb{G}_m(\mathcal{J})$.

One way:

$$\begin{tikzcd} & \widetilde{V} \arrow[d, "\text{quot by } \mu_m"] & \tilde{\infty} \arrow[l, hook'] \arrow[d] \\ X & V = X\setminus\{\infty\} \arrow[l, hook'] & \infty/\mu_m \arrow[l, hook'] \end{tikzcd}$$

Pullback

$$\begin{tikzcd} J_{\widetilde{V}} \arrow[r] \arrow[d] & J \arrow[d] \\ \widetilde{V} \arrow[r, "/\mu_m"'] & V \arrow[r, "a"] & \mathcal{C}_{\mathcal{L}} \end{tikzcd}$$

Glue $\begin{matrix} J_a \\ \downarrow \\ \operatorname{Spec}\hat{\mathcal{O}} \end{matrix}$ & $\begin{matrix} J_{a,V} \\ \downarrow \\ V \end{matrix}$ $= \mu_m$-descent from $J_{\widetilde{V}}$. to get J_X

Alternatively: $a: X \longrightarrow \mathcal{C}_{\mathcal{L}}$. a section.

Cartesian diag: commutative.

$$\begin{tikzcd} J_a \arrow[r] \arrow[d] & J_{\mathcal{L}} = \mathcal{J} \times_{\mathbb{G}_m^{dil}} \mathrm{Tot}(\mathcal{L}^\times) \arrow[d] \\ X \arrow[r, "a"] & \mathcal{C}_{\mathcal{L}} = \mathcal{C} \times_{\mathbb{G}_m^{dil}} \mathrm{Tot}(\mathcal{L}^\times). \end{tikzcd}$$

Define global $\mathcal{P}_a$ = moduli stack of J_a-torsors over X.

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A $\mathcal{J}_a$-torsor (Q_U, Q_V, τ) : $\begin{cases} Q_U : \mathcal{J}_a|_U\text{-torsor over } U = X \setminus \infty \\ Q_V : \mu_m\text{-equiv } \tilde{\mathcal{J}}_a\text{-torsor over } \tilde{V} \\ (\rightsquigarrow \mathcal{J}_a|_{U\cap V}\text{-torsor } Q^{6}_{U\cap V} \text{ on } U\cap V \text{ via descent}) \\ \tau : \text{isom of } \mathcal{J}_a|_{U\cap V}\text{-torsor} \\ \qquad \tau : Q_U \xrightarrow{\sim} Q^{6}_{U\cap V}. \end{cases}$

local analogue of $\mathcal{P}_a^{gl.}$

$x \in X - \{\infty\}$, $\quad P_{a,x}$ = moduli of $\mathcal{J}_a$-torsors over $\operatorname{Spec} \widehat{\mathcal{O}_x}$ together with a trivializ'n over $\operatorname{Spec} \widehat{K_x}$)

$$P_{a,x}(\mathbb{C}) = \mathcal{J}_a(\widehat{K_x}) / \mathcal{J}_a(\widehat{\mathcal{O}_x})$$

At $x = \infty$, $\quad P_{a,\infty} = \left(\mathcal{J}_a(F_m) / \mathcal{J}_a(\mathcal{O}_{F_m}) \right)^{\mu_m}.$

Hitchin fibers. $(G = GL_n)$

Fix $a \in \mathcal{A}$. $(G = GL_n)$

Spectral curve: $Y_a \subseteq \mathrm{Tot}(\mathcal{L}) = \mathrm{Spec}(\mathcal{O} \oplus \mathcal{L}^{-1} \oplus \mathcal{L}^{-2} \oplus \cdots)$

$= \mathrm{Spec}(\mathrm{Sym}(\mathcal{O} \oplus \mathcal{L}^{-1}))$

$$\begin{tikzcd} Y_a \arrow[dr, "\deg n"', "\pi"] & \mathrm{Tot}(\mathcal{L}) \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

$a_i \in H^0(X, \mathcal{L}^{\otimes d_i})$

map of coh. sheaves on X

$\ell_a : \mathcal{L}^{-n} \longrightarrow \pi_* \mathcal{O}_{\mathrm{Tot}(\mathcal{L})} = \mathcal{O} \oplus \mathcal{L}^{-1} \oplus \mathcal{L}^{-2} \oplus \cdots$

$((-1)^n a_n, (-1)^{n-1} a_{n-1}, \ldots, -a_1, 1, 0 \cdots)$

adjunction $\rightsquigarrow$

$\ell_a' : \pi^* \mathcal{L}^{-n} \longrightarrow \mathcal{O}_{\mathrm{Tot}(\mathcal{L})}$, Image $= J_a$.

$Y_a = \mathrm{Spec}\left((\mathcal{O} \oplus \mathcal{L}^{-1} \oplus \cdots)/\mathcal{J}_a\right)$ ideal sheaf

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Trivilize $\mathcal{L}$ on $\mathcal{U} \subseteq X$ open.

coordinate y.

$$Y_a|_{\mathcal{U}} \subseteq \mathcal{U} \times \mathbb{A}^1$$

giving by $y^n - y^{n-1}a_1 + \cdots + (-1)^n a_n = 0$

prop: For $a \in A_{\mathcal{L}}^{\heartsuit}$,

torsion free coh. $\mathcal{O}_{Y_a}$-modules that're generically of rk 1

$$\mathcal{M}_a^{Hit} \cong \overline{\mathrm{Pic}}(Y_a)$$

isom. as stacks

- When Y_a smooth. $\mathrm{Pic}(Y_a) = \overline{\mathrm{Pic}}(Y_a)$.

Proof of prop:

Given: $(E, \phi : E \to E \otimes \mathcal{L})$

Think $\phi : \mathcal{L}^{-1} \to \mathcal{E}nd(E)$

or $\phi : \mathrm{Sym}(\mathcal{L}^{-1}) \to \mathcal{E}nd(E)$

$\Rightarrow$ E is a sheaf of mod over $\mathrm{Sym}(\mathcal{L}^{-1})$

Let $\tilde{E} :=$ corresp. Coh. sheaf on $\mathrm{Spec}\,\mathrm{Sym}(\mathcal{L}^{-1}) = \mathrm{Tot}(\mathcal{L})$

$\Rightarrow$ $\tilde{E}$ is supported on $Y_a \subseteq \mathrm{Tot}(\mathcal{L})$

(Since Hamilton Cayley thm $\Rightarrow$ $\forall$ matrix A satisfies its own char eqn).

$\pi_* \tilde{E} = E \Rightarrow \tilde{E}$ torsion free

Now check: $\tilde{E}_y$ dim 1, for $y \in \pi^{-1}(x)$, $x \in X$ generic pt.
fiber

21.1

Fix an isom. $\mathcal{L}_x \xrightarrow{\sim} \mathbb{C}$

$\Rightarrow \quad \phi_x : E_x \to E_x$ an operator

$y \longleftrightarrow$ an eigenvalue λ_y of ϕ_x.

$\tilde{E}_y = \mathrm{Coker}(\phi_x - \lambda_y)$

x generic $\Rightarrow$ λ_y multiplicity 1 $\Rightarrow$ $\dim \tilde{E}_y = 1$.

Inverse Construction:

$$\begin{array}{c} \tilde{E} \\ \downarrow \\ Y_a \\ \downarrow \pi \\ X \end{array}$$

$E := \pi_* \tilde{E}$

Canonical $\pi^* \mathcal{L}^{-1} \to \mathcal{O}_{\mathrm{Tot}(\mathcal{L})} \to \mathcal{O}_{Y_a}$

$\rightsquigarrow \mathcal{O}_{Y_a} \to \pi^* \mathcal{L}$.

$\overset{\otimes \tilde{E}}{\rightsquigarrow} \tilde{E} \to \tilde{E} \otimes \pi^* \mathcal{L}$

$\overset{\pi_*}{\rightsquigarrow} E \to E \otimes \mathcal{L}$.

$G = GL_n$

$\Rightarrow \operatorname{Pic}(Y_a) \curvearrowright \mathcal{M}^{Hit}_a = \overline{\operatorname{Pic}}(Y_a)$

<u>Claim</u>: $\mathcal{P}_a^{glob} = \operatorname{Pic}(Y_a)$

J_L : regular group scheme of centralizer.

$$\begin{tikzcd} \pi: Y_a \arrow[d] & \supseteq \pi^{-1}(x) \arrow[d] & & J_L \arrow[d] \\ X & \ni x \arrow[rr, "a"] & & \mathcal{C}_{X,L} \end{tikzcd}$$

The group $(J_a)_x$ = The group of invertible functions on $\pi^{-1}(x)$

$\mathcal{U} \subseteq X$ open $\quad \begin{matrix} J_a \\ \downarrow \\ \mathcal{U} \end{matrix}$ Same as an invertible function on $Y_a|_{\mathcal{U}}$.

$\Rightarrow$ J_a-torsor on X = G_m-torsor on Y_a

$\Rightarrow \mathcal{P}_a^{global} = \operatorname{Pic}(Y_a)$

Prop 6.3.7 [OY]: Let $\nu > 0$, addmissible slope.
$\nu = d/m$

Assume: The cocharacter $d \cdot \xi^{\vee} \in X_*(\mathbb{T}^{cool})$ lifts to $X_*(\mathbb{T})$.

(1) $\exists$ a surjective map:

$$Sp_{\mathbb{P},\nu} \times_{P^{loc}} P^{global} = \mathcal{M}^{\heartsuit}_{\mathbb{P},\nu} \longrightarrow H^1(\mu_m, \mathbb{T})$$

fibers are homeomorphic to $[Sp_{\mathbb{P},\nu} / \tilde{S}]$

(2) If ν is elliptic then:
($\Rightarrow$ $H^1(\mu_m, \mathbb{T}) = 0$ Tate-complex calculation)

we have a homeomorphism over

$$[Sp_{\mathbb{P},\nu} / \tilde{S}] \cong \mathcal{M}^{\heartsuit}_{\mathbb{P},\nu}$$

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fiberwise equiv. under $\mathcal{P}_v^{loc} \rtimes \mathbb{G}_m(v)$

& $\mathcal{P}_v^{glo.} \rtimes \mathbb{G}_m(v)$.

① What's $\widetilde{S}$? $\left(1 \to \widetilde{S} \to \mathcal{P}_v^{loc} \to \mathcal{P}_v^{global} \longrightarrow H^1(\mu_m, \mathbb{T}) \to 1.\right)$ stacks.

$\widetilde{S}$ gp scheme over $\mathcal{C}(F)_v^{rs}$:

Define:

$\widetilde{S}_a :=$ the centralizer of γ in $G(F)^{\mathbb{G}_m(v)} \subseteq G_\gamma(F)$.

By Lemma 3.3.5 (3): $\widetilde{S}_a \cong \mathbb{T}^{\pi_a(\mu_m)}$, where $\pi_a : \hat{\mathbb{Z}}(1) \longrightarrow W$.

About the isom: For fixed γ, $\gamma(1) \in \mathfrak{g}^{r.s}$

$\mathbb{T}' = C_G(\gamma(1))$ is a max torus

& The principal grading $\underline{\Phi} : \mu_m \longrightarrow \mathbb{G}^{ad}$

normalize it

$\Rightarrow$ $\underline{\Phi}$ induces

$$\pi : \mu_m \longrightarrow N_{\mathbb{G}^{ad}}(T')/_{T'} = W'$$

$$\Rightarrow \tilde{S}_\alpha = C_{\mathbb{G}}(\gamma(1))^{\underline{\Phi}} = \mathbb{T}'^{\,\pi(\mu_m)} \cong \mathbb{T}^{\pi(\mu_m)}$$

Centralizer in $\mathbb{G}$ of both the grading $\underline{\Phi}$ & $\gamma(1)$.

Example 6.6.4:

$G = PGL_2$. $\qquad \gamma = \begin{pmatrix} 0 & 1 \\ t^d & 0 \end{pmatrix}$. $\qquad d>0$ odd integer.

$$\overset{\text{Conj}}{\sim} \begin{pmatrix} -t^{d/2} & 0 \\ 0 & t^{d/2} \end{pmatrix}.$$

$a = \chi(\gamma)$. $\qquad \nu = d/2$.

since: $s^d \cdot \begin{pmatrix} -t^{d/2} & 0 \\ 0 & t^{d/2} \end{pmatrix} = \begin{pmatrix} -(ts^2)^{d/2} & \\ & (ts^2)^{d/2} \end{pmatrix}$!

$\pi_1(PGL_2) = \pi_0(PGL_2(F)) = \mathbb{Z}/2$.

$\Rightarrow \quad Fl_G = G(F)/G(\mathcal{O})$ has two components

& $Fl_{G,a}$

Regular Centralizer gp

$J_a =$ Centralizer γ in G

$$= \left\{ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ t^d & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ t^d & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right.$$

$$\Updownarrow$$

$$\begin{bmatrix} bt^d & a \\ dt^d & c \end{bmatrix} = \begin{bmatrix} c & d \\ t^d a & t^d b \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{cases} bt^d = c \\ a = d \\ \cancel{dt^d = t^d a} \\ \cancel{c = t^d b} \end{cases} \Rightarrow g = \begin{bmatrix} a & b \\ bt^d & a \end{bmatrix}$$

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$$J_a(F) = \left\{ \begin{pmatrix} x & y \\ t^d y & x \end{pmatrix} \;\middle|\; x, y \in F, \quad x^2 - t^d y^2 \neq 0 \right\} \Big/ F^\times .$$

$$\pi : \mu_2 \longrightarrow W = S_2 \curvearrowright \underset{\substack{\cup \\ \left[\begin{smallmatrix} 1 & \\ & a \end{smallmatrix}\right]}}{T}$$

$$-1 \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & a^{-1} \end{bmatrix} \text{ in } PGL_2$$

$$\tilde{S}_a \cong T^{\mu_2} \cong \{\pm 1\} \longrightarrow J_a(F)$$

$$-1 \longmapsto \begin{bmatrix} 0 & 1 \\ t^d & 0 \end{bmatrix}.$$

$\tilde{S}_a \curvearrowright Sp_{G,a}$ by permutes the two componets.

$$\Rightarrow \quad Sp_{G,a} / \tilde{S}_a = (Sp_{G,a})^{\circ}.$$

Describe $\mathcal{M}_{G,a}$, $a = -\xi^d \in \Gamma(X, \mathcal{O}(d/2)^2) = \mathcal{A}$ 29

$\mathcal{L} = \mathcal{O}(n)$ (tr = 0 & we only have det)

$= \mathcal{O}(d/2)$

A point in $\mathcal{M}_{G,a} = \mathcal{M}_a / \mathrm{Pic}(X)$

$= \{ (V, \varphi) \mid \begin{array}{c} V \\ \downarrow \\ X \end{array}$ rank 2 v. bundle. $\varphi : V \to V \otimes \mathcal{O}(d/2)$

s.t. $\varphi^2 = \xi^2 \cdot \mathrm{id} : V \xrightarrow{\varphi} V \otimes \mathcal{O}(d/2) \xrightarrow{\varphi} V \otimes \mathcal{O}(d) \}$.

$/ \mathrm{Pic}(X)$

$\mathrm{Pic}(X) \curvearrowright \mathcal{M}_a$

$N \qquad (V, \varphi) \longmapsto (N \otimes V, \mathrm{id}_N \otimes \varphi)$.

Any $V = \mathcal{O}(a) \oplus \mathcal{O}(b)$

May assume $V = \mathcal{O} \oplus \mathcal{O}(n/2)$ (up to $\mathrm{Pic}(X)$-action)

$$\mathcal{O} \oplus \mathcal{O}(n/2) \xrightarrow{\varphi} \mathcal{O}(d/2) \oplus \mathcal{O}\left(\frac{n+d}{2}\right)$$

$$\varphi = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}.$$

$$\left\{\begin{array}{l} x \in \mathbb{C}[\xi, \eta]_d \\ y \in \mathbb{C}[\xi, \eta]_{d-n} \\ z \in \mathbb{C}[\xi, \eta]_{d+n} \\ x^2 + yz = \overbrace{\xi^d}^{\deg : 2d} \end{array}\right.$$

$\boxed{\begin{array}{l} \deg(\xi) = 2 \\ \deg(\eta) = 1 \end{array}}$

and $x^2 + yz = \xi^d$

$$\Updownarrow$$

$$\left(\begin{bmatrix} x & y \\ z & -x \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & -x \end{bmatrix} = \begin{bmatrix} x^2 + yz & 0 \\ 0 & zy + x^2 \end{bmatrix} = \xi^d I \right)$$

$\Rightarrow$ n must be odd. (?) & $1 \le n \le d$.

$$\begin{cases} \deg_\xi(x^2) < d \\ \deg_\xi(yz) < d. \end{cases}$$

Monomial like

$\deg(\xi^a \eta^b) = 2a + b = \begin{cases} \text{even}, & b \text{ even} \\ \text{odd} & b \text{ odd} \end{cases}$

n even $\Rightarrow \eta \mid x^2 + yz \Rightarrow \eta \mid x, y, z.$?

$\boxed{d=1} \Rightarrow x^2 + yz = \xi \Rightarrow x = \eta,\ \underbrace{y = \eta}_{\deg = 1-n},\ \underbrace{z = -\eta}_{\deg\ 1-n}.$

$\eta \cdot \begin{bmatrix} x' & y' \\ z' & -x' \end{bmatrix}$

$S_n :=$ the space of all such matrices

Fix x, $\Rightarrow$ finite many choices of y, z. up to scalar.

$\Rightarrow \dim S_n = \underset{?}{1} + \underbrace{\dim \mathbb{C}[\xi, \eta]_d}_{x \text{ lies}}$

$= 1 + \dfrac{(d+1)}{2}$

$= \dfrac{d+3}{2}$

Basis.

$\xi^{1/2}, \eta$

$\underbrace{(\xi^{1/2})^d, (\xi^{1/2})^{d-1}\eta, \ldots, (\xi^{1/2})^0 \eta^d}_{\text{half of the elts doesn't make sense.}}$

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$H_n := \mathrm{Aut}(\mathcal{O} \oplus \mathcal{O}(\frac{n}{2})) / \mathbb{G}_m \quad \circlearrowright \quad S_n$ by conj.

$$\Rightarrow \quad \begin{bmatrix} x & 0 \\ y & 1 \end{bmatrix} \qquad \begin{array}{l} x \in \mathbb{C}^\times \\ y \in \mathbb{C}[\xi, \eta]_n \end{array}$$

$$\dim(H_n) = \frac{n+3}{2} = 1 + \frac{n+1}{2}$$

$$\Rightarrow \quad M_a = \coprod_{\substack{1 \le n \le d \\ n \text{ odd}}} S_n / H_n \qquad \dim S_n / H_n = \frac{d-n}{2}.$$

stratificat.

Top. strata $\quad \underbrace{S_1 / H_1}_{\text{one top dim component!}} \leftarrow \dim \frac{d-1}{2}.$

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Example $G = SL_2$. $a = \chi(\gamma)$

$$= \chi\begin{pmatrix} 0 & 1 \\ t^d & 0 \end{pmatrix}.$$

$$Sp_{SL_2, a} \cong \left(Sp_{PGL_2, a}\right)^{\circ}$$

connected component.

But: $\mathcal{M}_{G, a} = \phi$.

Reason: $(V, \varphi) \in \mathcal{M}_{G,a} \Rightarrow V = \mathcal{O}(-n/2) \oplus \mathcal{O}(n/2)$.

SL_2-bundle.

$$\Rightarrow \quad \varphi = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$$

$$\begin{cases} x \in \mathbb{C}[\xi, \eta]_d \\ y \in \mathbb{C}[\xi, \eta]_{d-2n} \end{cases}$$

(34)

$$\begin{cases} z \in \mathbb{C}[\xi, \eta] & \deg + 2k_1 \\ x^2 + yz = \xi^d . \end{cases}$$

$$\begin{cases} d \text{ odd} \\ \xi \deg 2 \end{cases}$$

the $\deg_{\xi}(x^2) < d$

$\deg_{\xi}(yz) < d$

$\Rightarrow$ $x^2 + yz = \xi^d$ has <u>no solution</u>!

Assumption (6.9) <u>fails</u> in this case!

Step 1 : Define the map:

$$Sp_{\mathbb{P},v} \times_{\hat{P}_v^{loc}} P_v^{gl} \longrightarrow \mathcal{M}_{\mathbb{P},v} \quad \text{over } A_v^{\heartsuit}.$$

(a) The map: $e_0 : A_v \longrightarrow c(F)_v$ is an isomorphism.

[lemma 6.5.1].

$$\begin{array}{ccc} a & \longmapsto & a|_{\operatorname{spec}\hat{K}_0} \\ \cup\!\! \mid & & \cup\!\! \mid \\ A_v^{\heartsuit} & \xrightarrow{\cong} & c(F)_v^{rs} \end{array}$$

(b). Construct the Higgs bundle on V (§ 6.6.1) with invariants $(\mathcal{E}_V^{triv} \times A_v^{\heartsuit}, \varphi_V)$ over $A_v^{\heartsuit}$. s.t $a_V := a|_V$

where: $V = X \setminus \{0\}$.

no automorphism

$0 = [0\ 1]$

$\infty = [1, 0]$

automorphism μ_m

Let: $\mathcal{E}^{triv}_V = \mathcal{G}_V$: trivial $\mathcal{G}_V$-torsor over V.

$$\begin{tikzcd}
 & \tilde{V} \arrow[d, "/\mu_m"] & \tilde{\infty} \arrow[l, hook'] \arrow[d] \\
X & V \arrow[l, hook'] & \infty/\mu_m \arrow[l, hook']
\end{tikzcd}$$

$V = X - \{s\}$

$\mu_m \curvearrowright G$ - trivially.

$H^1(\mu_m, G) = \mathrm{Hom}(\mu_m, G)$

$= \mathbb{Z}/m\mathbb{Z}$

A Higgs field on $\mathcal{E}^{triv}_V$:

$b.\ \tilde{V} \longrightarrow \mathfrak{g}$ together with 1-cocycle

$\varepsilon: \mu_m \to G$

s.t: $\forall \zeta \in \mu_m$,

$$b(\zeta v) = \zeta^d \, \mathrm{Ad}\underset{G}{(\varepsilon(\zeta))}\, b(v).$$

Recall:

$\mathcal{L}$ line bundle on $\tilde{V}$, with μ_m-equivariant structure.

classified by a class in $H^1(\mu_m, \mathbb{G}_m)$

The class $[\mathcal{L}|_V]$ is d/m. $\Rightarrow$ $\exists$ μ_m-equiv. isom $\mathcal{L}|_{\tilde{V}} \longrightarrow \tilde{V} \times \mathbb{A}^1$

with $\mathcal{L}|_V = \mathcal{L}(\nu)$ and $d = \nu$,

s.t. μ_m action on $\mathbb{A}^1$ via d-th power

Choose the Higgs field φ_V^a: 138

Take σ to be: $\tilde{V} \xrightarrow{a_V} \mathfrak{C} \xrightarrow[\uparrow]{\kappa} \mathfrak{g}$

Kostant section.

where:

$a_V = a|_V : \tilde{V} \longrightarrow \mathfrak{C}$.

$\uparrow$

μ_m-equivariant, where $\mu_m \curvearrowright \mathfrak{C}$ by

$\xi \cdot \mathfrak{C} \longrightarrow \mathfrak{C}$

$c \longmapsto \xi^{d} \cdot c$.

Take the 1-cocycle to be:

$\mu_m \longrightarrow G$

$\varepsilon(\xi) = \xi^{-\lambda}$, $\lambda \in X_*(\mathbb{T})$ lifting of $d\rho^\vee \in X_*(T^{ad})$.

Define the map:

$$\begin{array}{ccc} \beta_{\mathbb{P},v} : & Sp_{\mathbb{P},v} & \longrightarrow \mathcal{M}_{\mathbb{P},v} \\ & \rotatebox{90}{$\in$} & \\ & g\mathbb{P} & \end{array}$$

$g\mathbb{P} \longmapsto$ • a $\mathbb{P}$-Higg bundle $(\mathcal{E}_0, \varphi_0)$ over $\operatorname{Spec} \widehat{\mathcal{O}}_0$.

trivial $\mathbb{P}$-torsor

$\varphi_0 = \operatorname{Ad}(g^{-1})\, \kappa(a)$
$\in \mathcal{E}_0 \overset{\mathbb{P}}{\times} \operatorname{Lie} \mathbb{P}$

• An isomorphism $(\mathcal{E}_0, \varphi_0)\big|_{\operatorname{Spec} \widehat{K}_0} \xrightarrow[\operatorname{Ad}(g)]{\sim} (G, \kappa(a))$.

Now: $(\mathcal{E}_V^{triv}, \varphi_V^{a})\big|_{\operatorname{Spec} \widehat{K}_0} = (G, \kappa(a))$

$\Rightarrow$ Glue $(\mathcal{E}_0, \varphi_0)$ with $(\mathcal{E}_v^{triv}, \varphi_v^a)$ along $\mathrm{Spec}\,\mathcal{O}_v$ $\mathrm{Spec}\,F_v$.

$\Rightarrow$ Get a map $\boxed{\beta_{\mathbb{P},a} : Sp_{\mathbb{P},a} \longrightarrow \mathcal{M}_{\mathbb{P},a}.}$

$a \in \mathcal{A}_v^{\heartsuit}$.

By construction, $P_v^{loc} \longrightarrow \mathcal{P}_v^{gl}$ over $\mathfrak{C}(F)_v^{r.s} \xrightarrow{\simeq} \mathcal{A}_v^{\heartsuit}$.

$$\Rightarrow \quad Sp_{\mathbb{P},v} \underset{P_v^{loc}}{\times} \mathcal{P}_v^{gl} \longrightarrow \mathcal{M}_{\mathbb{P},v}.$$

<u>Step 2</u>: The short exact seq of stacks

$$1 \to \tilde{S} \to P_v^{loc} \longrightarrow \mathcal{P}_v^{gl} \longrightarrow H^1(\mathbb{G}_m, \mathbb{T}) \to 1.$$

The map: $\mathcal{P}^{loc} \longrightarrow \mathcal{P}^{gl}$:

gluing a J_a-torsor over Spec $\hat{\mathcal{O}}$ with a trivial $J_{a,V}$-torsor over V

$V \longrightarrow [\mathfrak{c}^{rs}]$ lift to

$\tilde{V} \longrightarrow \mathfrak{t}^{rs}_{\bar{\nu}}$, $\bar{\nu} \in \mathbb{Q}/\mathbb{Z}$.

The regular centralizer $J_{a,V} = \tilde{V} \overset{\mu_m}{\times} \mathbb{T}$

§3.3.7: use this lifting.

$\mu_m \curvearrowright \mathbb{T}$ by

$\pi: \mu_m \to W \curvearrowright \mathbb{T}$

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$\Rightarrow$ $J_{a,V}$-torsor on V $\xleftrightarrow{1:1}$ μ_m-equivrt Π-torsors over $\widetilde{V}$.

Any Π-torsor on $\widetilde{V}$ is trivial, &

μ_m-equiv. str on trivial Π-torsor

$\in$

$H^1(\mu_m, \Pi)$.

$\Rightarrow$ Gives $\mathcal{P}^{loc} \longrightarrow H^1(\mu_m, \Pi)$

If a J_a-torsor on X has trivial class in $H^1(\mu_m, \Pi)$,

$\Rightarrow$ it comes from gluing a J_a-torsor on Spec $\widehat{\mathcal{O}}_0$.

$\Rightarrow$ lies in the image of P_a^{loc} $\Rightarrow$ $H^1(\mu_m, \Pi) = \text{Cokernel}$.

$$\ker\left(\mathcal{P}^{loc}_a \longrightarrow \mathcal{P}^{gl}_a\right) = \text{automorphism gp of such a } \bar{J}_a\text{-torsor}$$

$$= T^{\pi(\mathcal{U}_m)}$$

$$\cong \tilde{S}_a .$$

∎